Please hand in your solutions before the lecture on Monday.

Problem 25. Check that a sequence $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ of sheaves of abelian groups on a topological space X is exact iff the induced sequence of stalks $\mathscr{F}_p \to \mathscr{G}_p \to \mathscr{H}_p$ is exact at every point $p \in X$. Show that a morphism of sheaves is determined uniquely by the induced morphisms on stalks, and the former is invertible iff the latter are all invertible. Does any of these properties hold for presheaves?

Problem 26. Show that every soft sheaf $\mathscr{F}$ on a paracompact Hausdorff space X satisfies

$$H^i(X, \mathscr{F}) \; = \; 0 \quad \text{for all} \quad i \; > \; 0$$

by adapting the proof of the analogous statement for flabby sheaves from the lecture.

Problem 27. Let $f : X_1 \to X_2$ a continuous map. For sheaves $\mathscr{F}_i \in \mathrm{Sh}(X_i)$ of abelian groups we define direct and inverse image presheaves by sending an open subset $U_i \subseteq X_i$ to

$$\begin{aligned} f_*(\mathscr{F}_1)(U_2) \; &:= \; \mathscr{F}_1(f^{-1}(U_2)), \\ f^\sharp(\mathscr{F}_2)(U_1) \; &:= \; \varinjlim_{V} \; \mathscr{F}_2(V), \end{aligned}$$

where in the second line $V \subseteq X_2$ runs over all open subsets with $f(U_1) \subseteq V$ and the limit is defined as in the case of stalks (take the disjoint union over all such open subsets and divide out by the equivalence relation which identifies local sections if they agree on some common open subset containing $f(U_1)$).

(a) Show that $f_*(\mathscr{F}_1)$ is a sheaf.

(b) Show that $f^\sharp(\mathscr{F}_2)$ is a presheaf, but find an example where it is not a sheaf.

(c) Show that for the associated sheaf $f^{-1}(\mathscr{F}_2) := (f^\sharp(\mathscr{F}_2))^s$ we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Sh}(X_1)}\big(f^{-1}(\mathscr{F}_2), \mathscr{F}_1\big) \; \simeq \; \mathrm{Hom}_{\mathrm{Sh}(X_2)}\big(\mathscr{F}_2, f_*(\mathscr{F}_1)\big).$$

Bonus problem. Put $X = \mathbb{C} \setminus \Sigma$ for a finite set $\Sigma \subset \mathbb{C}$. Compute $H^1(X, \mathbb{Z}_X)$ by taking Cech cohomology for a cover consisting of two open subsets.