Please hand in your solutions before the lecture on Monday.

Problem 28. Let X be a triangulated surface. For each vertex v let $U_v \subseteq X$ be the interior of the union of all faces containing v. Show that Cech cohomology of $\mathbb{R}_X$ on the open cover $\mathscr{U} = \{U_v \mid v \text{ vertex}\}$ is dual to simplicial homology:

$$\check{H}^n(\mathscr{U}, \mathbb{R}_X) \simeq \operatorname{Hom}_{\mathbb{R}}\big(H_n(X, \mathbb{R}), \mathbb{R}\big) \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Problem 29. Let X be a complex manifold.

(a) Show that a holomorphic line bundle L on X is isomorphic to the trivial line bundle iff it has a nowhere vanishing holomorphic section.

(b) Deduce that $L \otimes L^*$ is always trivial and that the set of isomorphism classes of holomorphic line bundles on X forms a group. We denote it by $\operatorname{Pic}(X)$.

(c) Show that $\operatorname{Pic}(X) \simeq H^1(X, \mathscr{O}_X^*)$.

Problem 30. Let X be a Riemann surface. Show that the map $d\log : f(z) \mapsto \frac{f'(z)}{f(z)} dz$ gives an exact sequence

$$0 \longrightarrow \mathbb{C}_X^* \longrightarrow \mathscr{O}_X^* \xrightarrow{d\log} \Omega_X^1 \longrightarrow 0$$

where Ω_X^1 denotes the sheaf of holomorphic 1-forms. If $X = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, find the cokernel of

$$d\log : \quad H^0(X, \mathscr{O}_X^*) \longrightarrow H^0(X, \Omega_X^1).$$

Bonus Problem (if you know singular cohomology). Let X be a locally contractible space and

$$\mathscr{F}^n = \text{sheaf associated to the presheaf } U \mapsto C^n(U, \mathbb{Z}),$$

where $C^n(U, \mathbb{Z})$ denotes the group of singular n-cochains on $U \subseteq X$. Show that we have a flabby resolution $0 \to \mathbb{Z}_X \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots$ and deduce that singular cohomology coincides with the sheaf cohomology of the constant sheaf.