Please hand in your solutions before the lecture on Monday.

Problem 31. Let $X = \mathbb{C}/\Lambda$ be a torus, and fix a point $p \in X$. Use the Weierstras function $\wp_\Lambda$ from problem set 7 and the Riemann-Roch theorem to compute the dimensions

$$\dim_{\mathbb{C}} H^i\big(X, \mathscr{O}_X(np)\big) \quad \text{for} \quad i = 0, 1 \quad \text{and all} \quad n \in \mathbb{Z}.$$

Problem 32. Use the Riemann-Roch theorem to show that every compact Riemann surface of genus $g = 2$ is hyperelliptic and comes from an algebraic curve defined by an equation

$$y^2 = x(x-1)(x-a)(x-b)(x-c) \quad \text{with pairwise distinct} \quad a, b, c \in \mathbb{C} \setminus \{0, 1\}.$$

Problem 33. Clarify the relation between smooth and holomorphic tangent bundles:

(a) Let V be a complex vector space and denote by $V_{\mathbb{R}}$ the same additive group seen as a real vector space. Show that there is a natural isomorphism of complex vector spaces

$$\varphi : \quad \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}} \xrightarrow{\sim} V \oplus \overline{V}$$

where $\overline{V}$ denotes the complex vector space with the same additive group as V but scalar multiplication

$$\mathbb{C} \times \overline{V} \longrightarrow \overline{V}, \quad (z, v) \mapsto \overline{z} \cdot v.$$

Extend this result to smooth complex vector bundles on a smooth manifold.

(b) If X is a Riemann surface and $z = x + iy$ is a local coordinate, then dx, dy form a real basis for the fibers in the corresponding local trivialization of the cotangent bundle T_X^*. Express the isomorphism φ from (a) in this basis.

Bonus problem. Let $X = \mathbb{P}^1(\mathbb{C})$ and $\mathscr{L} = \mathscr{O}_X(d \cdot \infty)$ with $d \in \mathbb{Z}$. Using the Cech description for the standard open cover from the lecture, write down explicitly a nondegenerate bilinear form $\langle \cdot, \cdot \rangle : H^0(X, \omega_X \otimes \mathscr{L}^*) \otimes H^1(X, \mathscr{L}) \longrightarrow \mathbb{C}$.