Please hand in your solutions at the beginning of the problem class on Friday.

Problem 1. (a) Show that the Cauchy-Riemann equations for $f = u + iv : U \to \mathbb{C}$ can be written as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \phi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \phi}$$

in polar coordinates (r, ϕ) on any sufficiently small open subset $U \subset \mathbb{C} \setminus \{0\}$.

(b) Deduce that on any such open there exists a holomorphic function $f : U \to \mathbb{C}$ such that $\exp(f(z)) = z$. Fixing any such function near $z = 1$, compute its analytic continuation along the two paths

$$\gamma_\pm : \quad [0, \pi] \longrightarrow \mathbb{C} \setminus \{0\}, \quad t \mapsto \exp(\pm it).$$

Problem 2. Show that every 1-dimensional complex torus $\mathbb{C}/\Lambda$ is isomorphic to one of the form

$$X_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau \quad \text{for some} \quad \tau \in \mathbb{H} = \big\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\big\}.$$

Show furthermore that

$$X_\sigma \simeq X_\tau \quad \text{if} \quad \sigma = \frac{a\tau + b}{c\tau + d} \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Problem 3. In the lecture we have defined a topological manifold as a topological space which is

(a) Hausdorff,

(b) second countable, and

(c) covered by charts homeomorphic to open subsets in $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Show by examples that none of these three conditions follows from the other two.

Please hand in your solutions at the beginning of the problem class on Friday.

Problem 4. Let X be a Riemann surface and $f : X \setminus \{p\} \longrightarrow \mathbb{C}$ a holomorphic function on the complement of a point p. Show that the following three conditions are equivalent:

(a) f does not extend to a meromorphic function on X,

(b) for every open neighborhood $U \subseteq X$ of p, the image $f(U \setminus \{p\})$ is dense in $\mathbb{C}$,

(c) there exists a sequence of points $p_n \in X \setminus \{p\}$ converging to p such that the sequence

$$\big(f(p_n)\big)_{n\in\mathbb{N}} \quad \text{does not converge in} \quad \mathbb{P}^1(\mathbb{C}).$$

Problem 5. Determine the group $\mathrm{Aut}(\mathbb{C})$ of biholomorphic automorphisms of the complex plane. Hint: For any such automorphism $f \in \mathrm{Aut}(\mathbb{C})$ consider the Laurent series of $z \mapsto f(1/z)$ on $\mathbb{C} \setminus \{0\}$, and apply the previous exercise.

Problem 6. Let X be a Hausdorff topological space and $\overline{X} = X \cup \{\infty\}$ its disjoint union with another point.

(a) Show that $\overline{X}$ becomes a compact topological space by declaring $U \subseteq \overline{X}$ to be open iff

- either $U \subseteq X$ is an open part of the original topological space,
- or $U = \overline{X} \setminus K$ is the complement of a compact subset $K \subseteq X$.

(b) Compute $\overline{X}$ for $X = \mathbb{R}^n$ and for $X = (0,1) \cup (2,3)$.

Bonus Problem (optional). If you would like to review complex differentiability,

(a) show that $f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}, f(z) = \frac{z^5}{|z|^4}$ extends to a partially differentiable function over the origin and satisfies the Cauchy-Riemann equations there. Is the extended function holomorphic at the origin?

(b) recall that $g : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}, g(z) = z^2 \sin(1/z)$ extends differentiably over the origin but its derivative is not continuous there. What if you replace $\mathbb{R}$ by $\mathbb{C}$ and ask for complex differentiability?

Please hand in your solutions at the beginning of the problem class on Friday.

Problem 7. In class we have seen that any meromorphic function $f \in \mathscr{M}(\mathbb{P}^1(\mathbb{C}))$ is a rational function

$$f(z) \;=\; \frac{p(z)}{q(z)} \quad \text{for some} \quad p, q \;\in\; \mathbb{C}[z], \quad q \neq 0.$$

Use this to determine the group of biholomorphic automorphisms $\operatorname{Aut}(\mathbb{P}^1(\mathbb{C}))$.

Problem 8. Which of the following Riemann surfaces are isomorphic? In each case write down a biholomorphic map or explain why such a map cannot exist:

$$\begin{aligned} D &= \{z \in \mathbb{C} \mid |z| < 1\}, & A &= \{z \in \mathbb{C} \mid \tfrac{1}{2} < |z| < 1\}, & &\mathbb{C}, \\ D^* &= D \setminus \{0\}, & \mathbb{H} &= \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}, & &\mathbb{P}^1(\mathbb{C}). \end{aligned}$$

Problem 9. Let $f_s : (X, x_0) \to (Y, y_0)$ for $s = 0, 1$ be continuous maps of pointed topological spaces. We say that the maps are *homotopic* if there is a continuous map

$$H: \quad [0,1] \times X \;\to\; Y \quad \text{with} \quad \begin{cases} H(s, -) \;=\; f_s & \text{for } s = 0, 1, \\ H(s, x_0) \;=\; y_0 & \text{for all } s \in [0,1]. \end{cases}$$

Show that this gives an equivalence relation on maps. If $HTop_*$ is the category whose objects are pointed topological spaces and whose morphisms are homotopy classes of maps, verify that $HTop_* \to Groups, (X, x_0) \mapsto \pi_1(X, x_0)$ is a well-defined functor. What is the relation between

(a) homeomorphisms,

(b) isomorphisms in the category $HTop_*$,

(c) continuous maps inducing an isomorphism on fundamental groups?

Bonus Problem (optional). If you feel uneasy about line integrals, reprove Liouville's theorem for bounded holomorphic $f : \mathbb{C} \to \mathbb{C}$ by looking at

$$\oint_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - a)(\zeta - b)}\, d\zeta \qquad \text{for} \quad a, b \in \mathbb{C} \quad \text{and} \quad R \to \infty.$$

Please hand in your solutions at the beginning of the problem class on Friday.

Problem 10. Let X be a topological space.

(a) Show that if $U_1, U_2 \subseteq X$ are simply connected open subsets with $U_1 \cap U_2$ pathconnected and nonempty, then $U_1 \cup U_2$ is simply connected.

(b) Deduce that the sphere $X = S^n$ is simply connected for $n \geq 2$.

(c) For $n = 2$, find a continuous curve $\gamma : [0,1] \twoheadrightarrow S^2$ covering the sphere.

Problem 11. Show that for $X \subseteq \mathbb{C}$ open, a holomorphic function $f : X \to \mathbb{C}$ is a local homeomorphism near a point $x_0 \in X$ iff $f'(x_0) \neq 0$. Apply this criterion to see that

$$\sin : \quad X \;=\; \mathbb{C} \setminus \left\{ \tfrac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \right\} \;\to\; Y \;=\; \mathbb{C} \setminus \{\pm 1\}$$

is a covering map. Let $u_\pm : [0,1] \to Y$ be the curves given by $t \mapsto \pm(e^{2\pi i t} - 1)$, and consider the composite

$$\gamma_+ \;=\; u_+ \cdot u_- \quad \text{and} \quad \gamma_- \;=\; u_- \cdot u_+.$$

Find $\tilde{\gamma}_\pm(1)$ for the unique continuous lifts $\tilde{\gamma}_\pm : [0,1] \to X$ of $\gamma_\pm$ with $\tilde{\gamma}_\pm(0) = 0$.

Problem 12. Let $p : X \longrightarrow Y$ be a covering map of finite degree between Riemann surfaces. Show that for any $h \in \mathscr{O}(X)$,

$$P_h(t) \quad := \prod_{x \in p^{-1}(y)} \big(t - h(x)\big) \;\in\; \mathscr{O}(Y)[t]$$

is a polynomial in t whose coefficients depend holomorphically on y. Deduce that the ring extension $\mathscr{O}(Y) \hookrightarrow \mathscr{O}(X), f \mapsto f \circ h$ is *integral*, i.e. any element of the bigger ring is a zero of a monic polynomial over the smaller ring.

Bonus problem. If $U \subseteq \mathbb{C}$ is a connected open subset and $\{f_n : U \to \mathbb{C}\}_{n \in \mathbb{N}}$ is a sequence of injective holomorphic functions converging uniformly on compact subsets to a function $f : U \to \mathbb{C}$, show that f is either injective or constant.

Please hand in your solutions at the beginning of the problem class on Friday.

Problem 13. Let $\Lambda, \Lambda' \subset \mathbb{C}$ be lattices.

(a) Show that any holomorphic map $f : \mathbb{C}/\Lambda' \to \mathbb{C}/\Lambda$ with $f(0) = 0$ comes from a linear map

$$\tilde{f} : \mathbb{C} \to \mathbb{C}, \; z \mapsto az \quad \text{for a unique } a \in \mathbb{C} \text{ with } a \cdot \Lambda' \subseteq \Lambda.$$

(b) If $\Lambda' = \Lambda$ and if $f \neq \pm\mathrm{id}$ is a non-trivial automorphism, show that a must be a root of unity with the property that $\Lambda = \mathbb{Z}\lambda \oplus \mathbb{Z}a\lambda$ for any non-zero lattice vector $\lambda \in \Lambda \setminus \{0\}$ of minimal length.

(c) Find all automorphisms of the complex tori

$$\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \quad \text{and} \quad \mathbb{C}/(\mathbb{Z} \oplus \rho\mathbb{Z}) \quad \text{for} \quad \rho = \exp\left(\tfrac{2\pi i}{3}\right),$$

and show that any torus as in part b is isomorphic to one of these two.

Problem 14. For $n \in \mathbb{N}$, show that $f_n : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}), z \mapsto z^n + z^{-n}$ is a branched Galois cover whose group of deck transformations is the dihedral group

$$D_{2n} = \left\langle \sigma, \rho \;\middle|\; \rho^n = \sigma^2 = \mathrm{id}, \;\; \sigma\rho\sigma = \rho^{-1} \right\rangle.$$

Problem 15. For each of the following two maps $f : Y = \mathbb{P}^1(\mathbb{C}) \to X = \mathbb{P}^1(\mathbb{C})$, find the ramification/branch locus

$$Y \supset R(f) \longrightarrow Br(f) \subset X$$

and the deck transformation group $\mathrm{Aut}(Y/X)$, and decide whether this branched cover is Galois:

(a) $f(z) = z^3 - 3z$,

(b) $f(z) = (z^2 + 1)^2$.

Bonus problem. If X is a Riemann surface and $G \to \mathrm{Aut}(X)$ is a freely discontinuous group action by biholomorphic automorphisms, check that the quotient $Y = X/G$ is again a Riemann surface. Determine $\pi_1(Y, y)$ when X is simply connected.

Please hand in your solutions at the beginning of the problem class on Friday.

Problem 16. Check that the affine algebraic curves

$$C = \{(x,y) \in \mathbb{C}^2 \mid y^2 = 3 + 10x^4 + 3x^8\}, \quad D = \{(z,w) \in \mathbb{C}^2 \mid w^2 = z^6 - 1\}$$

are both smooth. How many points must be added to make them compact Riemann surfaces? Show that

$$f: \quad C \setminus \{(\pm 1, \pm 4)\} \longrightarrow D, \quad (x,y) \mapsto \Big(\frac{1+x^2}{1-x^2}, \frac{2xy}{(1-x^2)^3} \Big)$$

is well-defined and extends to an unramified cover between these Riemann surfaces.

Problem 17. For $d \in \mathbb{N}$, find all $f \in \mathbb{C}[x]$ so that the polynomial $y^d - f(x) \in \mathbb{C}[x,y]$ is irreducible. In the irreducible case consider the algebraic curve

$$\{(x,y) \in \mathbb{C}^2 \mid y^d = f(x)\}.$$

Show that for the associated compact Riemann surface Y the map $x : Y \to \mathbb{P}^1(\mathbb{C})$ is a branched Galois cover. Find its deck transformation group, and decompose the field $\mathscr{M}(Y)$ of meromorphic functions into eigenspaces for the action of this group.

Problem 18. For $F \in \mathbb{C}[x,y,z]$ irreducible and homogenous of positive degree, show there is a unique compact Riemann surface with a holomorphic generically injective map $f : X \to \mathbb{P}^2(\mathbb{C})$ whose image is

$$f(X) \; = \; \{[x:y:z] \in \mathbb{P}^2(\mathbb{C}) \mid F(x,y,z) = 0\} \; \subset \; \mathbb{P}^2(\mathbb{C}).$$

Find this Riemann surface and the map f for the polynomial $F(x,y,z) = y^2 z - x^3$.

Bonus problem. Let $f \in \mathbb{C}[x,y]$. Viewing f as a polynomial over the field $K = \mathbb{C}(x)$, let $\alpha_1, \ldots, \alpha_n$ be its roots in an algebraic closure of K, including multiplicities. Show that

$$\Delta \; := \; \prod_{i<j} (\alpha_i - \alpha_j)^2 \; \in \; \mathbb{C}[x]$$

and that for any $x_0 \in \mathbb{C}$, one has $\Delta(x_0) = 0$ iff $f(x_0, y) \in \mathbb{C}[y]$ has a multiple root.

Please hand in your solutions at the beginning of the problem class on Friday.

Problem 19. Let $F : Y \to X$ be a branched cover of degree n.

(a) Check that the following properties are equivalent:

- $[\mathscr{M}(Y) : \mathscr{M}(X)] = n$,
- there is a function $t \in \mathscr{M}(Y)$ that is injective on a general fiber $F^{-1}(x)$.

(b) Verify these properties when $F : Y \to \mathbb{C}$ is obtained by glueing two copies of the complex plane along the countably many slits $[2n, 2n+1]$ with $n \in \mathbb{Z}$.

(c) Can you extend the double cover in (b) to a cover of the Riemann sphere?

Problem 20. Let $\Lambda \subset \mathbb{C}$ be a lattice.

(a) Show that on any compact subset $K \subset \mathbb{C}$ the series

$$\wp_\Lambda(z) \;=\; \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \Big(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \Big)$$

can be made uniformly convergent by removing finitely many summands, and that $\wp_\Lambda \in \mathscr{M}(\mathbb{C})$ is a well-defined meromorphic function with poles only in Λ.

(b) Look at derivatives to see that $\wp_\Lambda$ is Λ-periodic. Using that $\wp_\Lambda(z) = \wp_\Lambda(-z)$ is an *even* function of z, determine its multiplicity at every point.

Problem 21. Let $\Lambda \subset \mathbb{C}$ be a lattice and $f \in \mathscr{M}(\mathbb{C})$ a Λ-periodic function.

(a) If $f(z) = f(-z)$ is an even function of z, show that

- for $f^{-1}(\infty) \subseteq \Lambda$ one has $f = P(\wp_\Lambda)$ for some polynomial $P \in \mathbb{C}[x]$,
- in any case one has $f = R(\wp_\Lambda)$ for a rational function $R \in \mathbb{C}(x)$.

(b) In general, show that

$$f \;=\; R(\wp_\Lambda) + S(\wp_\Lambda) \cdot \wp_\Lambda' \quad \text{for rational functions} \quad R, S \;\in\; \mathbb{C}(x).$$

(c) Find R and S when $f(z) = (\wp_\Lambda'(z))^2$ and explain what this says about $\mathbb{C}/\Lambda$.

Next Friday we celebrate Grothendieck's 90th birthday at the BMS, so there will be no problem class. Please hand in your solutions before the lecture on Monday.

Problem 22. Two simple applications of the Riemann-Hurwitz formula:

(a) Let $f : X \to X$ be a branched cover of degree $d > 1$ from a compact Riemann surface to itself. What can you say about the genus of the surface and the number of ramification points? Find examples for all cases.

(b) For $n \in \mathbb{N}$, compute the genus of the compact Riemann surface attached to the algebraic curve $\{(x, y) \in \mathbb{C}^2 \mid x^n + y^n = 1\}$. Can you generalize this?

Problem 23. Let X be a Riemann surface, and take a finite group $G \subset \mathrm{Aut}(X)$ of biholomorphic automorphisms.

(a) Define the structure of a Riemann surface on the topological space $Y = X/G$.

Hint: First divide out the action of the stabilizer $G_p = \{g \in G \mid gp = p\}$ in some neighborhood of $p \in X$. Show that if z is a local coordinate centered at p then a local coordinate on the quotient is given by $\prod_{g \in G_p} g^*(z)$.

(b) If X is compact, show that

$$\chi(X) \;=\; |G| \cdot \chi(Y) - \sum_{p \in X} \big(|G_p| - 1 \big).$$

(c) Deduce that if a compact Riemann surface of genus two has an automorphism of prime order p, then $p \in \{2, 3, 5\}$. Look at hyperelliptic Riemann surfaces to find examples where these three primes actually occur.

Problem 24. Let X be a Riemann surface. For open subsets $U \subseteq X$ and $a \neq b \in X$ put

$$\begin{aligned}
\mathscr{E}(U) &= \mathscr{O}_X^*(U)/\exp(\mathscr{O}_X(U)),\\
\mathscr{F}(U) &= \{\, f \in \mathscr{O}_X(U) \mid f(a) = 0 \ \text{ if } \ a \in U \,\},\\
\mathscr{G}(U) &= \{\, f \in \mathscr{O}_X(U) \mid f(a) = f(b) \ \text{ if } \ a, b \in U \,\},\\
\mathscr{H}(U) &= \{\, \text{bounded continuous functions } f : U \to \mathbb{C} \,\}.
\end{aligned}$$

Which of these presheaves are sheaves? For the others, what is the associated sheaf?

Please hand in your solutions before the lecture on Monday.

Problem 25. Check that a sequence $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ of sheaves of abelian groups on a topological space X is exact iff the induced sequence of stalks $\mathscr{F}_p \to \mathscr{G}_p \to \mathscr{H}_p$ is exact at every point $p \in X$. Show that a morphism of sheaves is determined uniquely by the induced morphisms on stalks, and the former is invertible iff the latter are all invertible. Does any of these properties hold for presheaves?

Problem 26. Show that every soft sheaf $\mathscr{F}$ on a paracompact Hausdorff space X satisfies

$$H^i(X, \mathscr{F}) \;=\; 0 \quad \text{for all} \quad i \;>\; 0$$

by adapting the proof of the analogous statement for flabby sheaves from the lecture.

Problem 27. Let $f : X_1 \to X_2$ a continuous map. For sheaves $\mathscr{F}_i \in \mathrm{Sh}(X_i)$ of abelian groups we define direct and inverse image presheaves by sending an open subset $U_i \subseteq X_i$ to

$$\begin{aligned} f_*(\mathscr{F}_1)(U_2) \;&:=\; \mathscr{F}_1(f^{-1}(U_2)), \\ f^\sharp(\mathscr{F}_2)(U_1) \;&:=\; \varinjlim_{V} \; \mathscr{F}_2(V), \end{aligned}$$

where in the second line $V \subseteq X_2$ runs over all open subsets with $f(U_1) \subseteq V$ and the limit is defined as in the case of stalks (take the disjoint union over all such open subsets and divide out by the equivalence relation which identifies local sections if they agree on some common open subset containing $f(U_1)$).

(a) Show that $f_*(\mathscr{F}_1)$ is a sheaf.

(b) Show that $f^\sharp(\mathscr{F}_2)$ is a presheaf, but find an example where it is not a sheaf.

(c) Show that for the associated sheaf $f^{-1}(\mathscr{F}_2) := (f^\sharp(\mathscr{F}_2))^s$ we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Sh}(X_1)}\big(f^{-1}(\mathscr{F}_2), \mathscr{F}_1\big) \;\simeq\; \mathrm{Hom}_{\mathrm{Sh}(X_2)}\big(\mathscr{F}_2, f_*(\mathscr{F}_1)\big).$$

Bonus problem. Put $X = \mathbb{C} \setminus \Sigma$ for a finite set $\Sigma \subset \mathbb{C}$. Compute $H^1(X, \mathbb{Z}_X)$ by taking Cech cohomology for a cover consisting of two open subsets.

Please hand in your solutions before the lecture on Monday.

Problem 28. Let X be a triangulated surface. For each vertex v let $U_v \subseteq X$ be the interior of the union of all faces containing v. Show that Cech cohomology of $\mathbb{R}_X$ on the open cover $\mathscr{U} = \{U_v \mid v \text{ vertex}\}$ is dual to simplicial homology:

$$\check{H}^n(\mathscr{U}, \mathbb{R}_X) \simeq \operatorname{Hom}_{\mathbb{R}}\big(H_n(X, \mathbb{R}), \mathbb{R}\big) \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Problem 29. Let X be a complex manifold.

(a) Show that a holomorphic line bundle L on X is isomorphic to the trivial line bundle iff it has a nowhere vanishing holomorphic section.

(b) Deduce that $L \otimes L^*$ is always trivial and that the set of isomorphism classes of holomorphic line bundles on X forms a group. We denote it by $\operatorname{Pic}(X)$.

(c) Show that $\operatorname{Pic}(X) \simeq H^1(X, \mathscr{O}_X^*)$.

Problem 30. Let X be a Riemann surface. Show that the map $d\log : f(z) \mapsto \frac{f'(z)}{f(z)} dz$ gives an exact sequence

$$0 \longrightarrow \mathbb{C}_X^* \longrightarrow \mathscr{O}_X^* \xrightarrow{d\log} \Omega_X^1 \longrightarrow 0$$

where Ω_X^1 denotes the sheaf of holomorphic 1-forms. If $X = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, find the cokernel of

$$d\log : \quad H^0(X, \mathscr{O}_X^*) \longrightarrow H^0(X, \Omega_X^1).$$

Bonus Problem (if you know singular cohomology). Let X be a locally contractible space and

$$\mathscr{F}^n = \text{sheaf associated to the presheaf } U \mapsto C^n(U, \mathbb{Z}),$$

where $C^n(U, \mathbb{Z})$ denotes the group of singular n-cochains on $U \subseteq X$. Show that we have a flabby resolution $0 \to \mathbb{Z}_X \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots$ and deduce that singular cohomology coincides with the sheaf cohomology of the constant sheaf.

Please hand in your solutions before the lecture on Monday.

Problem 31. Let $X = \mathbb{C}/\Lambda$ be a torus, and fix a point $p \in X$. Use the Weierstras function $\wp_\Lambda$ from problem set 7 and the Riemann-Roch theorem to compute the dimensions

$$\dim_{\mathbb{C}} H^i\big(X, \mathscr{O}_X(np)\big) \quad \text{for} \quad i = 0, 1 \quad \text{and all} \quad n \in \mathbb{Z}.$$

Problem 32. Use the Riemann-Roch theorem to show that every compact Riemann surface of genus $g = 2$ is hyperelliptic and comes from an algebraic curve defined by an equation

$$y^2 = x(x-1)(x-a)(x-b)(x-c) \quad \text{with pairwise distinct} \quad a, b, c \in \mathbb{C} \setminus \{0, 1\}.$$

Problem 33. Clarify the relation between smooth and holomorphic tangent bundles:

(a) Let V be a complex vector space and denote by $V_{\mathbb{R}}$ the same additive group seen as a real vector space. Show that there is a natural isomorphism of complex vector spaces

$$\varphi : \quad \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}} \xrightarrow{\sim} V \oplus \overline{V}$$

where $\overline{V}$ denotes the complex vector space with the same additive group as V but scalar multiplication

$$\mathbb{C} \times \overline{V} \longrightarrow \overline{V}, \quad (z, v) \mapsto \overline{z} \cdot v.$$

Extend this result to smooth complex vector bundles on a smooth manifold.

(b) If X is a Riemann surface and $z = x + iy$ is a local coordinate, then dx, dy form a real basis for the fibers in the corresponding local trivialization of the cotangent bundle T_X^*. Express the isomorphism φ from (a) in this basis.

Bonus problem. Let $X = \mathbb{P}^1(\mathbb{C})$ and $\mathscr{L} = \mathscr{O}_X(d \cdot \infty)$ with $d \in \mathbb{Z}$. Using the Cech description for the standard open cover from the lecture, write down explicitly a nondegenerate bilinear form $\langle \cdot, \cdot \rangle : H^0(X, \omega_X \otimes \mathscr{L}^*) \otimes H^1(X, \mathscr{L}) \longrightarrow \mathbb{C}$.