On the error term of the prime geodesic theorem

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1 Introduction

1.1. Let $\mathbb{H}$ denote the upper half plane, $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ any Fuchsian group of the first kind and $M := \Gamma \backslash \mathbb{H}$ the corresponding quotient space. We denote by ℓ_γ the hyperbolic length of the closed geodesic determined by a hyperbolic element $\gamma \in \Gamma$. The prime geodesic counting function $\pi_M(u)$ $(u > 1)$ is defined as the number of inconjugate, primitive, hyperbolic elements $\gamma \in \Gamma$ such that $e^{\ell_\gamma} < u$. The prime geodesic theorem then states the estimate (see [3], [5], [6] and the references therein)

$$\pi_M(u) = \sum_{0 \leq \lambda_{M,j} < 1/4} \mathrm{li}(u^{s_{M,j}}) + O_M\left(u^{3/4}(\log u)^{-1/2}\right),$$

where $0 = \lambda_{M,0} < \lambda_{M,1} < \ldots$ are the eigenvalues of the hyperbolic Laplacian acting on the space of smooth functions on M and

$$s_{M,j} := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{M,j}}\ .$$

As indicated above the implied constant in the error term depends on M. The aim of this paper is to study this dependence, if M is a finite degree cover of a fixed base space $M_0 := \Gamma_0 \backslash \mathbb{H}$, i.e., Γ is a subgroup of finite index in the Fuchsian group of the first kind Γ_0. Under these assumptions the main result given in Theorem 3.4 states that, if C_{M_0} is a positive constant implied by the prime geodesic theorem on M_0, the quantity

$$C_M = [\Gamma_0 : \Gamma] \cdot C_{M_0}$$

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can be taken as the corresponding constant in the prime geodesic theorem for M.

1.2. The paper is organized as follows: In section 2, after fixing some notation, we recall the definitions of Selberg zeta functions, of heat kernels and of hyperbolic heat traces, as studied in [11], attached to M. We then recall from [11], p. 656, that the logarithmic derivative of the Selberg zeta function can be realized as a Gauss transform of the hyperbolic heat trace; if M is compact, this formula is sometimes referred to as McKean's formula, where the hyperbolic heat trace in this case is the trace of the heat kernel minus the volume of M times the heat kernel of the upper half plane $\mathbb{H}$ evaluated at distance zero. After recalling the definition of the prime geodesic counting function, we conclude this section by reminding the reader of the statement of the prime geodesic theorem.
In section 3, we make the additional assumption that M is a finite cover of some fixed quotient space $M_0 = \Gamma_0 \backslash \mathbb{H}$. We then extend the Artin formalism for the Selberg zeta functions under consideration (cf., e.g., [14], [15], [16], [17]) to the hyperbolic heat traces and the prime geodesic counting functions involved. This enables us to prove our main result in Theorem 3.4 using a generalized version of Weyl's law as given in Lemma 3.3.

1.3. As an independent problem, further analysis of the error term in the prime geodesic theorem, and the prime number theorems in general, are of interest in analytic number theory. In our work we were lead to the problem of understanding the error term in the prime geodesic theorem in our study of bounds for special values of the Selberg zeta function of Riemann surfaces (see [8]). Specifically, our bounds for the special values under consideration are effective and explicit up to knowing the implicit constant in the error term of the prime geodesic theorem, hence, the necessity for the work presented in this article. Going further, the method of proof we develop here will apply to a larger class of zeta functions, where one has the analogue of the McKean formula (which holds in wide generality) and where one has an Artin formalism. We leave the application of the method of proof in the present paper to more general settings to the interested reader.

2 Preliminaries

2.1. Riemann surfaces. Let $\mathbb{H}$ denote the upper half plane, $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ any Fuchsian group of the first kind and $M := \Gamma \backslash \mathbb{H}$ the associated quotient space, which can be given the structure of a hyperbolic Riemann surface. We denote by p_M, resp. n_M the number of cusps, resp. elliptic fixed points of M and by g_M the genus of the compact Riemann surface obtained from M by adding the cusps; we note that $p_M = 0$, if M is compact. We define the total branching order b_M of M by

$$b_M := \sum_{k=1}^{n_M} (m_k - 1)\,,$$

where m_k is the order of the corresponding elliptic element; we note that $b_M = 0$, if there are no elliptic elements in Γ. We recall that M is equipped with the complete hyperbolic metric of constant curvature -1, which is induced from the canonical hyperbolic metric on $\mathbb{H}$; we denote the corresponding hyperbolic volume element by $d\mu$. From [13], p. 42, we recall the formula

$$\frac{\mathrm{vol}(M)}{2\pi} = 2g_M - 2 + p_M + \sum_{k=1}^{n_M} \left(1 - \frac{1}{m_k}\right).$$

Using the well-known inequality $\mathrm{vol}(M)/2\pi \geq 1/42$ (cf. [13], p. 44), we easily derive from the above volume formula

$$p_M \leq \frac{85}{2\pi}\,\mathrm{vol}(M),\ n_M \leq \frac{85}{\pi}\,\mathrm{vol}(M).$$

2.2. The set $H(\Gamma)$. Let $H(\Gamma)$ denote a complete set of representatives of inconjugate, primitive, hyperbolic elements in Γ. For $\gamma \in H(\Gamma)$, we have $|\mathrm{tr}(\gamma)| > 2$; furthermore, we note that $\gamma \in H(\Gamma)$ generates its own centralizer in Γ. We denote by ℓ_γ the hyperbolic length of the closed geodesic determined by $\gamma \in H(\Gamma)$ on M; it is well-known that the equality

$$|\mathrm{tr}(\gamma)| = 2\cosh(\ell_\gamma/2)$$

holds.

2.3. Selberg zeta functions. For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, and ψ a finite dimensional, unitary representation of Γ of rank $\mathrm{rk}(\psi)$, the Selberg zeta function $Z_{M,\psi}(s)$ associated to M and ψ is defined via the Euler product expansion

$$Z_{M,\psi}(s) := \prod_{\gamma \in H(\Gamma)} Z_{\gamma,\psi}(s),$$

where the local factors $Z_{\gamma,\psi}(s)$ are given by

$$Z_{\gamma,\psi}(s) := \prod_{n=0}^{\infty} \det\left(1 - \psi(\gamma) e^{-(s+n)\ell_\gamma}\right).$$

If $\psi = \mathbf{1}$ is the trivial representation, we simply write $Z_M(s)$, resp. $Z_\gamma(s)$, instead of $Z_{M,\mathbf{1}}(s)$, resp. $Z_{\gamma,\mathbf{1}}(s)$. The Selberg zeta function $Z_{M,\psi}(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian (cf. [6], p. 498); furthermore, $Z_{M,\psi}(s)$ satisfies a functional equation. For our purposes, it suffices to know that the Selberg zeta function $Z_{M,\psi}(s)$ has a simple zero at $s = 1$, if $\psi = \mathbf{1}$, and is holomorphic and non-zero at $s = 1$, if $\psi \neq \mathbf{1}$.

2.4. Heat kernels. The heat kernel $K_{\mathbb{H}}(t;\tilde{x},\tilde{y})$ on $\mathbb{H}$ ($t\in\mathbb{R}_{>0}$; $\tilde{x},\tilde{y}\in\mathbb{H}$; $\tilde{x}\neq\tilde{y}$) is given by the formula

$$K_{\mathbb{H}}(t;\tilde{x},\tilde{y})=K_{\mathbb{H}}(t;\rho):=\frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int\limits_{\rho}^{\infty}\frac{re^{-r^2/4t}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,dr,$$

where $\rho:=d_{\mathbb{H}}(\tilde{x},\tilde{y})$ denotes the hyperbolic distance between $\tilde{x}$ and $\tilde{y}$ (cf. [4], p. 246); if $\tilde{x}=\tilde{y}$, i.e., $\rho=0$, the previous formula can be shown to equal

$$K_{\mathbb{H}}(t;\tilde{x},\tilde{x})=K_{\mathbb{H}}(t;0)=\frac{1}{2\pi}\int\limits_{0}^{\infty}e^{-(r^2+1/4)t}\,r\tanh(\pi r)\,dr.$$

The heat kernel $K_{M,\psi}(t;x,y)$ associated to M and ψ ($t\in\mathbb{R}_{>0}$; $x,y\in M$), resp. the hyperbolic heat kernel $HK_{M,\psi}(t;x)$ associated to M and ψ ($t\in\mathbb{R}_{>0}$; $x\in M$), are now defined by summing over the elements of Γ, resp. the hyperbolic elements of Γ, namely (cf. [5], p. 347),

$$\begin{aligned}K_{M,\psi}(t;x,y) &:= \sum_{\gamma\in\Gamma}\psi(\gamma)\,K_{\mathbb{H}}(t;\tilde{x},\gamma(\tilde{y}))\,,\ \text{resp.}\\ HK_{M,\psi}(t;x) &:= \sum_{\substack{\gamma\in\Gamma\\ |\mathrm{tr}(\gamma)|>2}}\psi(\gamma)K_{\mathbb{H}}(t;\tilde{x},\gamma(\tilde{x}));\end{aligned}$$

here $\tilde{x}$, resp. $\tilde{y}$ are (any) lifts of x, resp. y to $\mathbb{H}$. If $\psi=\mathbf{1}$ is the trivial representation, we simply write $K_M(t;x,y)$, resp. $HK_M(t;x)$, instead of $K_{M,\mathbf{1}}(t;x,y)$, resp. $HK_{M,\mathbf{1}}(t;x)$.

2.5. Hyperbolic heat traces. If M is non-compact, the hyperbolic heat kernel $K_{M,\psi}(t;x,y)$ is not of trace class whereas the hyperbolic heat kernel $HK_{M,\psi}(t;x)$ is integrable with respect to x. With this in mind, we define as in [11] the hyperbolic heat trace $HTrK_{M,\psi}(t)$ associated to M and ψ ($t\in\mathbb{R}_{>0}$) by the integral

$$HTrK_{M,\psi}(t):=\int\limits_{M}\mathrm{tr}\left(HK_{M,\psi}(t;x)\right)\,d\mu(x).$$

In analogy to the factorization of the Selberg zeta function $Z_{M,\psi}(s)$ into local factors, there is a decomposition of the hyperbolic heat trace $HTrK_{M,\psi}(t)$ into local constituents. Namely, we have

$$HTrK_{M,\psi}(t)=\sum_{\gamma\in H(\Gamma)}HTrK_{\gamma,\psi}(t),$$

where $HTrK_{\gamma,\psi}(t)$ is explicitly given by the expression

$$HTrK_{\gamma,\psi}(t) = \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=1}^{\infty} \frac{\mathrm{tr}(\psi(\gamma))\,\ell_\gamma}{e^{n\ell_\gamma/2} - e^{-n\ell_\gamma/2}} e^{-(n\ell_\gamma)^2/4t}$$

(cf. [5], p. 349; [6], p. 207). If $\psi = \mathbf{1}$ is the trivial representation, we write $HTrK_M(t)$, resp. $HTrK_\gamma(t)$, instead of $HTrK_{M,\mathbf{1}}(t)$, resp. $HTrK_{\gamma,\mathbf{1}}(t)$.
A spectral theoretical realization of the hyperbolic heat trace $HTrK_{M,\psi}(t)$ is obtained by an application of the Selberg trace formula. For this purpose, we let $\{\lambda_{M,\psi}\}$ denote the eigenvalues of the hyperbolic Laplacian acting on the space of smooth, global sections of the flat vector bundle associated to the representation ψ. There is only a finite number of eigenvalues $\lambda_{M,\psi}$ satisfying $0 \leq \lambda_{M,\psi} < 1/4$; as usual, we label these eigenvalues by $0 = \lambda_{M,\psi,0} < \lambda_{M,\psi,1} \leq \lambda_{M,\psi,2} \leq \ldots < 1/4$. Furthermore, there is a measure $d\mu_{M,\psi}$ defined on $\mathbb{R}$, which describes the distribution of the eigenvalues $\lambda_{M,\psi} \geq 1/4$, in particular of those eigenvalues constituting the continuous spectrum (cf. [6], p. 315). With these notations, we recall the formula

$$\begin{aligned} &HTrK_{M,\psi}(t) + \mathrm{rk}(\psi) \cdot \mathrm{vol}(M) \cdot K_{\mathbb{H}}(t;0) = \\ &\sum_{0 \leq \lambda_{M,\psi,j} < 1/4} e^{-\lambda_{M,\psi,j}\, t} + \int_0^\infty e^{-(r^2+1/4)t}\, d\mu_{M,\psi}(r). \end{aligned} \tag{1}$$

If M is compact, the measure $d\mu_{M,\psi}(r)$ is concentrated with mass one in the points of the form $r = \sqrt{\lambda_{M,\psi} - 1/4}$ with $\lambda_{M,\psi} \geq 1/4$. Again, to simplify notations, we write λ_M, resp. $\lambda_{M,j}$ instead of $\lambda_{M,\mathbf{1}}$, resp. $\lambda_{M,\mathbf{1},j}$, if $\psi = \mathbf{1}$ is the trivial representation.

2.6. McKean formulae. As a next ingredient, we recall the McKean formulae for the Selberg zeta function and its local factors (cf. [12] and [11], p. 656). For this purpose, we let $s \in \mathbb{C}$ with $\mathrm{Re}(s(s-1)) > 0$. Then, the McKean formula for the local factor $Z_{\gamma,\psi}(s)$ states the equality

$$\frac{Z'_{\gamma,\psi}}{Z_{\gamma,\psi}}(s) = (2s-1) \int_0^\infty HTrK_{\gamma,\psi}(t) e^{-s(s-1)t}\, dt.$$

Accordingly, the McKean formula for $Z_{M,\psi}(s)$ becomes

$$\frac{Z'_{M,\psi}}{Z_{M,\psi}}(s) = \sum_{\gamma \in H(\Gamma)} \frac{Z'_{\gamma,\psi}}{Z_{\gamma,\psi}}(s) = (2s-1) \int_0^\infty HTrK_{M,\psi}(t) e^{-s(s-1)t}\, dt.$$

2.7. Small eigenvalues. A small eigenvalue of the hyperbolic Laplacian is any eigenvalue, which is less than $1/4$. For any $a \geq 0$ and $a < b \leq 1/4$, we let

$$N_{M,\psi;a,b} := \#\{\lambda_{M,\psi,j} \,|\, a \leq \lambda_{M,\psi,j} < b\}.$$

To simplify notation, we supress the index ψ, if $\psi = \mathbf{1}$. From [19], p. 1620 (cf. also [2]), we recall the upper bound

$$N_{M;0,1/4} \leq 4g_M + 2p_M + 2n_M - 2.$$

From this and the estimate

$$\begin{aligned} \frac{\mathrm{vol}(M)}{\pi} &= 4g_M - 4 + 2p_M + 2\sum_{k=1}^{n_M}\left(1 - \frac{1}{m_k}\right) \\ &\geq 4g_M - 4 + 2p_M + n_M, \end{aligned}$$

we easily deduce the alternative upper bound

$$N_{M;0,1/4} \leq \frac{128}{\pi}\,\mathrm{vol}(M),$$

using also $\mathrm{vol}(M) \geq 1/42$ and $n_M \leq 85/\pi \cdot \mathrm{vol}(M)$.

2.8. The prime geodesic counting function. Consider the function

$$\pi_{M,\psi}(u) := \sum_{\substack{\gamma\in H(\Gamma)\\ e^{\ell_\gamma}<u}} \mathrm{tr}(\psi(\gamma)),$$

which is defined for $u \in \mathbb{R}_{>1}$. If $\psi = \mathbf{1}$ is the trivial representation, we simply write $\pi_M(u)$ instead of $\pi_{M,\mathbf{1}}(u)$, which is just the number of inconjugate, primitive, hyperbolic elements of Γ such that the corresponding geodesics have length less than $\log u$. Letting now $\ell_0 > 0$ denote a real number, which is smaller than the length of the shortest closed geodesic on M, and introducing the function

$$f(u,t) := \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=1}^{\infty} \frac{\log u}{u^{n/2} - u^{-n/2}} e^{-(n\log u)^2/4t},$$

we immediately derive from the decomposition of the hyperbolic heat trace into its local constituents the formula

$$HTrK_{M,\psi}(t) = \int_{e^{\ell_0}}^{\infty} f(u,t)\,d\pi_{M,\psi}(u).$$

2.9. The prime geodesic theorem. For any small eigenvalue $\lambda_{M,\psi,j}$ we put

$$s_{M,\psi,j} := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{M,\psi,j}}\,,$$

and note that $1/2 < s_{M,\psi,j} \leq 1$. Introducing the integral logarithm

$$\mathrm{li}(u^{s_{M,\psi,j}}) := \int_{2}^{u^{s_{M,\psi,j}}} \frac{d\xi}{\log \xi}\,,$$

the prime geodesic theorem states

$$\pi_{M,\psi}(u) = \sum_{0\le\lambda_{M,\psi,j}<1/4} \mathrm{li}(u^{s_{M,\psi,j}}) + O\left(u^{3/4}(\log u)^{-1/2}\right)$$

for $u > 2$, where the implied constant depends solely on M and ψ (cf. [7], p. 474).

3 An estimate for the error term for covers

3.1. Artin formalism. We fix a Fuchsian group $\Gamma_0 \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind and denote the associated quotient space $\Gamma_0\backslash\mathbb{H}$ by M_0. We then consider subgroups $\Gamma \subseteq \Gamma_0$, which have finite index in Γ_0; the associated quotient spaces $M = \Gamma\backslash\mathbb{H}$ are then finite covers of M_0 of degree $[\Gamma_0 : \Gamma]$. In [14], [15], [16], [17] it is shown among other things that the Selberg zeta functions associated to M and M_0 are related by the formula

$$\frac{Z'_M}{Z_M}(s) = \sum_{\psi} \mathrm{mult}(\psi)\,\frac{Z'_{M_0,\psi}}{Z_{M_0,\psi}}(s)\,, \tag{2}$$

where the sum is taken over all irreducible representations ψ occuring with multiplicity $\mathrm{mult}(\psi)$ in the representation $\mathrm{ind}_{\Gamma}^{\Gamma_0}(\mathbf{1})$, which is the representation induced by the trivial representation $\mathbf{1}$ on Γ to Γ_0. If the cover is normal, meaning Γ is a normal subgroup of Γ_0, then $\mathrm{mult}(\psi) = \mathrm{rk}(\psi)$. We refer to equation (2) as the additive Artin formalism satisfied by the logarithmic derivatives of the Selberg zeta functions $Z_M(s)$ and $Z_{M_0,\psi}(s)$ (cf. also [10]). More generally, we will say that a system of functions f_M and $f_{M_0,\psi}$ associated to M and M_0,ψ, respectively, satisfies the additive Artin formalism, if the functions in question are related by an equation analogous to (2).

3.2. Lemma. The following system of functions associated to M and M_0,ψ satisfies the additive Artin formalism:
(a) The logarithmic derivatives of the Selberg zeta functions $Z_M(s)$, $Z_{M_0,\psi}(s)$.
(b) The hyperbolic heat traces $HTrK_M(t)$, $HTrK_{M_0,\psi}(t)$.
(c) The prime geodesic counting functions $\pi_M(u)$, $\pi_{M_0,\psi}(u)$.
(d) The integral logarithms $\sum_{0\le\lambda_{M,j}<1/4}\mathrm{li}(u^{s_{M,j}})$, $\sum_{0\le\lambda_{M_0,\psi,j}<1/4}\mathrm{li}(u^{s_{M_0,\psi,j}})$.
(e) The counting functions $\#\{0 \le \lambda_{M,j} < T\}$, $\#\{0 \le \lambda_{M_0,\psi,j} < T\}$.

Proof. (a) The proof of (a) follows immediately from Venkov's work discussed in 3.1.

(b) To prove part (b), we introduce the new variable $w = (s-1/2)^2$, $\mathrm{Re}(w) > 1/4$, in the McKean formulae for the functions $Z_M(s)$, $Z_{M_0,\psi}(s)$. Then, we use part (a) to conclude that the functions

$$\frac{d}{dw}\left(\log Z_M(s)\right) = \int_0^\infty e^{-wt} e^{t/4} HTrK_M(t)\,dt,$$

$$\frac{d}{dw}\left(\log Z_{M_0,\psi}(s)\right) = \int_0^\infty e^{-wt} e^{t/4} HTrK_{M_0,\psi}(t)\,dt$$

satisfy the additive Artin formalism provided that $\mathrm{Re}(w) > 1/4$. An application of an inverse Laplace transform, justified by [18], Theorem 7.6a, then shows that the hyperbolic heat traces under consideration themselves satisfy the additive Artin formalism.

(c) For a test function f defined on $\mathbb{R}$ and satisfying some mild Fourier conditions as described in [9], III.1, the Mellin transform $\mathbf{M}f$ exists; here, we need to consider test functions f_u depending on an additional, real parameter $u > 1$. By [9], III.2, we then know that the integrals

$$V_M(f_u) := \frac{1}{2\pi i}\int_{\mathcal{C}} \mathbf{M}f_u(s)\frac{Z'_M}{Z_M}(s)\,ds\,,$$

$$V_{M_0,\psi}(f_u) := \frac{1}{2\pi i}\int_{\mathcal{C}} \mathbf{M}f_u(s)\frac{Z'_{M_0,\psi}}{Z_{M_0,\psi}}(s)\,ds\,,$$

taken along a suitably chosen contour $\mathcal{C}$ in the complex plane, exist. From part (a) we conclude that the functions V_M and $V_{M_0,\psi}$ satisfy the additive Artin formalism.

In [9], the Γ-factor occuring in the functional equation for Z_M, resp. $Z_{M_0,\psi}$, is denoted by Φ_M, resp. $\Phi_{M_0,\psi}$. Taking the logarithmic derivative of the functional equations for Z_M and $Z_{M_0,\psi}$, we then deduce again from part (a) that the functions Φ'_M/Φ_M and $\Phi'_{M_0,\psi}/\Phi_{M_0,\psi}$ satisfy the additive Artin formalism. This shows, as before, that the functions

$$W_M(f_u) := \frac{1}{2\pi i}\int_{\mathcal{C}} \mathbf{M}f_u(s)\frac{\Phi'_M}{\Phi_M}(s)\,ds\,,$$

$$W_{M_0,\psi}(f_u) := \frac{1}{2\pi i}\int_{\mathcal{C}} \mathbf{M}f_u(s)\frac{\Phi'_{M_0,\psi}}{\Phi_{M_0,\psi}}(s)\,ds\,,$$

with the contour $\mathcal{C}$ chosen as above, also satisfy the additive Artin formalism. The explicit formulas given in [9], III.2 (Theorem 2.2), now show that the functions

$$\Pi_M(f_u) \quad := \quad \sum_{\substack{n\geq 1\\ \gamma\in H(\Gamma)}} \frac{1}{n}\cdot\frac{n\ell_\gamma}{e^{n\ell_\gamma/2}-e^{-n\ell_\gamma/2}}\, f_u(e^{n\ell_\gamma}),$$

$$\Pi_{M_0,\psi}(f_u) \quad := \quad \sum_{\substack{n\geq 1\\ \gamma\in H(\Gamma_0)}} \frac{1}{n}\cdot\frac{\mathrm{tr}(\psi(\gamma))\, n\ell_\gamma}{e^{n\ell_\gamma/2}-e^{-n\ell_\gamma/2}}\, f_u(e^{n\ell_\gamma})$$

also satisfy the additive Artin formalism. Finally, we choose our test function f_u as follows: For $u>1, u\neq e^{n\ell_\gamma}$, we let f_u be a smooth, compactly supported function on $\mathbb{R}$ satisfying $f_u(x)=f_u(1/x)$ for $x\in\mathbb{R}$, $f_u(x)=0$ for $x>u$ and

$$f_u(x)=\frac{e^{n\ell_\gamma/2}-e^{-n\ell_\gamma/2}}{n\ell_\gamma}$$

for the discrete set of values of x of the form $x=e^{n\ell_\gamma}, 1<x<u$. With this choice in mind, we find

$$\Pi_M(f_u)=\sum_{\substack{n\geq 1,\gamma\in H(\Gamma)\\ e^{n\ell_\gamma}<u}}\frac{1}{n}\ ,\ \ \Pi_{M_0,\psi}(f_u)=\sum_{\substack{n\geq 1,\gamma\in H(\Gamma_0)\\ e^{n\ell_\gamma}<u}}\frac{\mathrm{tr}(\psi(\gamma))}{n}\,,$$

which proves the additive Artin formalism for the prime geodesic counting functions $\pi_M(u)$ and $\pi_{M_0,\psi}(u)$ as long as u is smaller than any $e^{2\ell_\gamma}$. Inductively, we conclude in this way that $\pi_M(u)$ and $\pi_{M_0,\psi}(u)$ satisfy the additive Artin formalism, provided that $\log u$ is not an integral multiple of the length of an element of $H(\Gamma_0)$.

(d) We let $\mathcal{C}$ denote a closed contour in the complex plane, which encloses the line segment $[1/2+\eta,1]\subset\mathbb{R}$ for sufficiently small $\eta>0$ and passes arbitrarily close to the left of $1/2+\eta$ and arbitrarily close to the right of 1. By multiplying both sides of equation (2) with the integral logarithm $\mathrm{li}(u^s)$ and then integrating along the contour $\mathcal{C}$, we obtain after division by $2\pi i$

$$\sum_{s\in\mathrm{int}(\mathcal{C})}\mathrm{res}\left(\mathrm{li}(u^s)\,\frac{Z_M'}{Z_M}(s)\right)=\frac{1}{2\pi i}\int_{\mathcal{C}}\mathrm{li}(u^s)\,\frac{Z_M'}{Z_M}(s)\,ds=$$

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\sum_{\psi}\mathrm{mult}(\psi)\,\mathrm{li}(u^s)\,\frac{Z_{M_0,\psi}'}{Z_{M_0,\psi}}(s)\,ds=$$

$$\sum_{\psi}\mathrm{mult}(\psi)\sum_{s\in\mathrm{int}(\mathcal{C})}\mathrm{res}\left(\mathrm{li}(u^s)\,\frac{Z_{M_0,\psi}'}{Z_{M_0,\psi}}(s)\right),$$

where $\mathrm{int}(\mathcal{C})$ denotes the interior of the set circumscribed by the contour $\mathcal{C}$. Since the logarithmic derivative $Z_M'/Z_M(s)$, resp. $Z_{M_0,\psi}'/Z_{M_0,\psi}(s)$, has poles of order one with residue one precisely at $s=s_{M,j}$, resp. at $s=s_{M_0,\psi,j}$, we end up with the formula

$$\sum_{0\leq\lambda_{M,j}<1/4}\mathrm{li}(u^{s_{M,j}})=\sum_{\psi}\mathrm{mult}(\psi)\sum_{0\leq\lambda_{M_0,\psi,j}<1/4}\mathrm{li}(u^{s_{M_0,\psi,j}}),$$

as claimed.

(e) To prove this claim, we proceed as in the proof of part (d), but replacing the function $\mathrm{li}(u^s)$ in the argument by the constant function 1. $\square$

3.3. Lemma. Let $M_0=\Gamma_0\backslash\mathbb{H}$ be as in 3.1 and ψ any finite dimensional, unitary representation of Γ_0. Then, there exists a constant $A_{M_0}>0$ depending only on M_0 such that the measure $d\mu_{M_0,\psi}$ described in formula (1) of section 2.5 satisfies

$$|\mu_{M_0,\psi}(r)|\leq A_{M_0}\cdot\mathrm{rk}(\psi)\cdot r^2$$

for any $r>0$.

Proof. The spectral theoretical realization of the hyperbolic heat trace as quoted in section 2.5, formula (1), shows in the case of the trivial representation $\psi=\mathbf{1}$ the existence of a positive function F_{M_0} depending on M_0 such that $t\cdot F_{M_0}(t)$ is bounded as t approaches zero and such that the inequalities

$$-F_{M_0}(t)\leq HTrK_{M_0}(t)\leq F_{M_0}(t)$$

hold for $t>0$. From the obvious bound $|\mathrm{tr}(\psi)|\leq\mathrm{rk}(\psi)$, we immediately derive the inequalities

$$-\mathrm{rk}(\psi)\,HTrK_{M_0}(t)\leq HTrK_{M_0,\psi}(t)\leq\mathrm{rk}(\psi)\,HTrK_{M_0}(t),$$

hence,

$$-F_{M_0}(t)\leq\frac{1}{\mathrm{rk}(\psi)}HTrK_{M_0,\psi}(t)\leq F_{M_0}(t)$$

for $t>0$. Furthermore, we obviously have

$$0\leq\frac{1}{\mathrm{rk}(\psi)}\sum_{0\leq\lambda_{M_0,\psi,j}<1/4}e^{-\lambda_{M_0,\psi,j}t}\leq\frac{N_{M_0,\psi;0,1/4}}{\mathrm{rk}(\psi)}.$$

Following the argument given in [4], p. 251 (Theorem 3), we find that Zograf's bound quoted in 2.7, extends to the case of arbitrary, finite dimensional, unitary representations ψ of Γ_0; therefore, there is an absolute constant $A>0$ such that the inequality

$$\frac{N_{M,\psi;0,1/4}}{\mathrm{rk}(\psi)}\leq A\cdot\mathrm{vol}(M_0)$$

holds. Substituting these inequalities into formula (1), we get for $t > 0$

$$
\begin{aligned}
&-F_{M_0}(t) + \mathrm{vol}(M_0) \cdot K_{\mathbb{H}}(t;0) - A \cdot \mathrm{vol}(M_0) \leq \\
&\frac{1}{\mathrm{rk}(\psi)} \int\limits_0^\infty e^{-(r^2+1/4)t}\, d\mu_{M_0,\psi}(r) \leq F_{M_0}(t) + \mathrm{vol}(M_0) \cdot K_{\mathbb{H}}(t;0),
\end{aligned}
$$

which can be rearranged into the form

$$
\begin{aligned}
&\mathrm{vol}(M_0) \cdot K_{\mathbb{H}}(t;0) \leq \\
&\frac{1}{\mathrm{rk}(\psi)} \int\limits_0^\infty e^{-(r^2+1/4)t}\, d\mu_{M_0,\psi}(r) + F_{M_0}(t) + A \cdot \mathrm{vol}(M_0) \leq \\
&2 \cdot F_{M_0}(t) + \mathrm{vol}(M_0) \cdot K_{\mathbb{H}}(t;0) + A \cdot \mathrm{vol}(M_0).
\end{aligned}
$$

We can now apply a Tauberian theorem (cf., e.g., [1], p. 94 or [9], p. 27) to the positive function

$$
\frac{1}{\mathrm{rk}(\psi)} \int\limits_0^\infty e^{-(r^2+1/4)t}\, d\mu_{M_0,\psi}(r) + F_{M_0}(t) + A \cdot \mathrm{vol}(M_0)
$$

from which the claim follows. □

3.4. Theorem. Let M be a finite cover of M_0 as in 3.1 and let C_{M_0} be a positive constant implied by the prime geodesic theorem for M_0. Then, we can take

$$
C_M = [\Gamma_0 : \Gamma] \cdot C_{M_0}
$$

as the corresponding constant in the prime geodesic theorem for M.

Proof. Following [3], chapter 9.6, in the compact case and formally the same argument in the non-compact setting, combined with an application of the Selberg trace formula, we are able to give the following refinement of the prime geodesic theorem stated in 2.9:

$$
\pi_{M_0,\psi}(u) = \sum_{0 \leq \lambda_{M_0,\psi,j} < 1/4} \mathrm{li}(u^{s_{M_0,\psi,j}}) + O_{M_0}\left(\mathrm{rk}(\psi) \cdot u^{3/4} (\log u)^{-1/2}\right);
$$

here the error term is obtained as in the discussion of [3], p. 260, using Weyl's law and taking into account our modification of Weyl's law established in Lemma 3.3. By means of the prime geodesic theorem for M and Lemma 3.2, we then conclude

$$O_M\left(u^{3/4}(\log u)^{-1/2}\right) = \pi_M(u) - \sum_{0\le\lambda_{M,j}<1/4} \mathrm{li}(u^{s_{M,j}}) =$$

$$\sum_\psi \mathrm{mult}(\psi)\left[\pi_{M_0,\psi}(u) - \sum_{0\le\lambda_{M_0,\psi,j}<1/4} \mathrm{li}(u^{s_{M_0,\psi,j}})\right] =$$

$$\sum_\psi \mathrm{mult}(\psi)\ O_{M_0}\left(\mathrm{rk}(\psi)\cdot u^{3/4}(\log u)^{-1/2}\right).$$

Now the claim of the theorem follows immediately recalling the relation

$$\sum_\psi \mathrm{mult}(\psi)\cdot\mathrm{rk}(\psi) = [\Gamma_0 : \Gamma].$$

□

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