Sup-norm bounds for automorphic forms in the cocompact case

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1 Introduction

1.1. In the theory of Riemann surfaces there are a number of natural metrics which arise from various points of view. In the setting of uniformization theory, every Riemann surface admits a universal cover which can be taken to be either the Riemann sphere $\mathbb{P}$, the complex plane $\mathbb{C}$, or the hyperbolic plane $\mathbb{H}$. Consequently, every Riemann surface admits a unique, up to scale, metric which is compatible with its complex structure and has constant curvature. On the other hand, in the setting of algebraic geometry, every compact Riemann surface can be embedded via the Abel-Jacobi map into its Jacobian variety. The Jacobian admits a canonically defined flat metric which can be pulled back to the Riemann surface, yielding the canonical metric.

1.2. In this article, we relate the canonical metric to the hyperbolic metric. More precisely, we estimate the canonical metric form in terms of the hyperbolic metric form for any compact Riemann surface M covering a fixed Riemann surface M_0 of finite degree. If $M = \Gamma\backslash\mathbb{H}$, where Γ is a discrete, cocompact, torsionfree subgroup of $\mathrm{PSL}_2(\mathbb{R})$, and $\{f_1, ..., f_{g_M}\}$ is an orthonormal basis of modular forms of weight 2 with respect to Γ, our main result shows

$$\sup_{z\in\mathbb{H}} \sum_{j=1}^{g_M} (\mathrm{Im} z)^2 |f_j(z)|^2 = O_{M_0}(1). \tag{1}$$

1.3. The proof of our main result comes from analyzing long time asymptotics of heat kernels. The idea of studying such asymptotics for heat kernels on forms

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in order to relate the canonical or, more generally, the Bergman metric to the hyperbolic metric comes from [4] and [7]. However, the results we state here go beyond those of [4] in two respects. First, our proof yields a bound which is uniform over $\mathbb{H}$, whereas in [4], the main result is pointwise, which then is extended to uniformity on compact subsets of $\mathbb{H}$. Second, our bounds extend to the non cocompact setting to yield pointwise results, which are uniform over compact subsets of $\mathbb{H}$, while [4] does not address this point though it is most likely possible without much effort. Also, the emphasis in the present article is to state the results as sup-norm bounds of automorphic forms rather than convergence results of sequences of metrics.

We also note that A. Abbes and E. Ullmo obtained in the non cocompact setting for the congruence subgroups $\Gamma_0(N)$ the estimate (cf. [1], Théorème B)

$$\sup_{z\in\mathbb{H}} \sum_{j=1}^{g_M} (\mathrm{Im} z)^2 |f_j(z)|^2 = O_\varepsilon(N^{2+\varepsilon})$$

for any $\varepsilon > 0$; here $M = \Gamma_0(N)\backslash\mathbb{H}$. It remains to be investigated, if the strong bound (1) can be transfered to the non cocompact situation. In forthcoming work, we obtain a precise relation for the difference between the canonical and the hyperbolic metric form in terms of the heat kernel on $\mathbb{H}$ and a certain counting function of the hyperbolic length spectrum. Applications to towers of cofinite subgroups in $\mathrm{PSL}_2(\mathbb{R})$ are also addressed.

2 Background material

2.1. Hyperbolic and canonical metrics. Let Γ be a discrete, cocompact, torsionfree subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy\}$. We let M be the quotient space $\Gamma\backslash\mathbb{H}$ and denote by g_M the genus of M.

We denote by μ_M^{hyp} the *hyperbolic metric* of M, which is compatible with the complex structure of M and has constant negative curvature equal to minus one. Locally, we have

$$\mu_M^{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{y^2},$$

where by abuse of notation, here and henceforth, we identify M locally with its universal cover $\mathbb{H}$. The *hyperbolic volume* $\mathrm{vol}_M^{\mathrm{hyp}}$ is known to be given by the formula

$$\mathrm{vol}_M^{\mathrm{hyp}} = 4\pi(g_M - 1).$$

We define the *scaled hyperbolic metric* μ_M^{shyp} by

$$\mu_M^{\mathrm{shyp}} := \frac{1}{\mathrm{vol}_M^{\mathrm{hyp}}}\, \mu_M^{\mathrm{hyp}},$$

which measures the volume of M to be one.
Furthermore, let $M_k(\Gamma)$ denote the $\mathbb{C}$-vector space of modular forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f, g\rangle = \int\limits_M f(z)\,\overline{g(z)}\,y^k\,\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}\,;$$

here $f, g \in M_k(\Gamma)$. By choosing an orthonormal basis $\{f_1, ..., f_{g_M}\}$ of $M_2(\Gamma)$ with respect to the Petersson inner product, we define the *canonical metric* μ_M^{can} of M by

$$\mu_M^{\mathrm{can}}(z) := \frac{1}{g_M}\cdot\frac{i}{2}\sum_{j=1}^{g_M}|f_j(z)|^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

We note that the canonical metric measures the volume of M to be one.

2.2. A differential operator. Define by $\mathcal{S}_1(\Gamma)$ the $\mathbb{C}$-vector space of functions g on $\mathbb{H}$, for which the integral

$$\int\limits_{\Gamma\backslash\mathbb{H}}|g(z)|^2\,\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}\,,$$

is bounded, and which satisfy the transformation formula

$$g\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{c\bar{z}+d}\right)\cdot g(z) \tag{2}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. From [5], p. 144, with $k=2$, or [6], p. 338, with $m=2$ (in both references the multiplier system v, as defined in [6], p. 332, is trivial) we recall the differential operator

$$\mathbf{D}_1 := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+2iy\frac{\partial}{\partial x}\,,$$

which acts on the vector space $\mathcal{S}_1(\Gamma)$. By results of [6] (namely: the definition on the bottom of p. 382; proposition 5.14 (a), (d) on p. 383; the first line of the proof on p. 383) it can be shown that a C^2-function $g \in \mathcal{S}_1(\Gamma)$ lies in the kernel of $\mathbf{D}_1$, if and only if $g(z)/y$ is holomorphic, i.e., $g(z)/y$ is a modular form of weight 2 with respect to Γ.

2.3. Heat kernels. Let $K_{\mathbb{H}}^{(1)}(t;z,w)$ ($t\in\mathbb{R}_{>0}$; $z,w\in\mathbb{H}$), resp. $K_M^{(1)}(t;z,w)$ ($t\in\mathbb{R}_{>0}$; $z,w\in M$) denote the heat kernel on $\mathbb{H}$, resp. M associated to the differential operator $\mathbf{D}_1$. From [5], p. 157 (see also [3], p. 539) one has explicit expressions for these heat kernels, which we now quote: For $t>0$ and $r>0$, let

$$g_1(t;r) := \Lambda_{1,0}(r) + \frac{e^{-t/4}}{(4\pi t)^{3/2}(\cosh(r)-1)}\int\limits_r^\infty\frac{ue^{-u^2/4t}}{\sqrt{2}\sqrt{\cosh(u)-\cosh(r)}}\,G(u;r)\,\mathrm{d}u\,,$$

where

$$\Lambda_{1,0}(r) := \frac{1}{2\pi(\cosh(r)+1)},$$

and

$$\begin{aligned} G(u;r) &:= e^{-2\theta}\left(\sqrt{2}\sinh(u/2)+\sqrt{\cosh(u)-\cosh(r)}\right)^2 \\ &+ e^{2\theta}\left(\sqrt{2}\sinh(u/2)-\sqrt{\cosh(u)-\cosh(r)}\right)^2 \end{aligned}$$

with $e^{\pm\theta}$ defined by

$$e^{\pm\theta}\sinh(r) := e^u - \cosh(r) \pm e^{u/2}\sqrt{2(\cosh(u)-\cosh(r))}\,.$$

We note that the limit $\lim_{r\to 0} g_1(t;r)$ exists and is given by

$$g_1(t;0) := \lim_{r\to 0} g_1(t;r) = \frac{1}{4\pi} + \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty \frac{2u\cosh(u)e^{-u^2/4t}}{\sqrt{2}\sqrt{\cosh(u)-1}}\,\mathrm{d}u\,.$$

The heat kernel on $\mathbb{H}$ associated to $\mathbf{D}_1$ is then given by

$$K_{\mathbb{H}}^{(1)}(t;z,w) := g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,w))\,,$$

where $\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(\cdot,\cdot)$ denotes the hyperbolic distance on $\mathbb{H}$. The function $\Lambda_{1,0}$ is defined in [5], p. 152 (see also [3], p. 539) in terms of the classical hypergeometric function.
Direct calculations show that $g_1(t;r)$ is positive for all $t>0$ and $r\geq 0$, and that the equality

$$\lim_{t\to\infty} g_1(t;0) = \frac{1}{4\pi}$$

holds. The heat kernel on M associated to $\mathbf{D}_1$ is given by the Poincaré series (see [5], p. 157, or [3], p. 540)

$$K_M^{(1)}(t;z,w) := \sum_{\gamma\in\Gamma}\left(\frac{c\bar{w}+d}{cw+d}\right)\left(\frac{z-\gamma(\bar{w})}{\gamma(w)-\bar{z}}\right) g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,\gamma(w)))\,,$$

where

$$\gamma(w) = \frac{aw+b}{cw+d} \text{ with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

note that in the above definition we have again identified M and $\mathbb{H}$ locally. We also recall from [5], p. 154, that the series defining $K_M^{(1)}(t;z,w)$ converges absolutely and uniformly for fixed $t\geq\delta$, for any $\delta>0$, and (z,w) ranging over $M\times M$.

2.4. Spectral expansions. Since M is compact, the spectrum of the differential operator $\mathbf{D}_1$ acting on the vector space $\mathcal{S}_1(\Gamma)$ is discrete. Let $\{g_{\lambda_j}\}_{\lambda_j \geq 0}$ be an orthonormal basis of $\mathcal{S}_1(\Gamma)$ consisting of eigenfunctions g_{λ_j} of $\mathbf{D}_1$ with eigenvalue $\lambda_j \geq 0$ (counting multiplicities). The heat kernel $K_M^{(1)}(t; z, w)$ now admits the spectral expansion

$$K_M^{(1)}(t; z, w) = \sum_{\lambda_j \geq 0} g_{\lambda_j}(z) \, \overline{g_{\lambda_j}(w)} \, e^{-\lambda_j t},$$

where the sum is taken over all eigenvalues $\lambda_j \geq 0$ (counting multiplicities). As discussed in [2], the above series converges absolutely and uniformly for fixed $t \geq \delta$, for any $\delta > 0$, and any $(z, w) \in M \times M$.

For our purposes, there are three main points from the spectral expansion of $K_M^{(1)}(t; z, w)$, which play an important role in the sequel. First, the function $K_M^{(1)}(t; z, z)$ is a well-defined, Γ-invariant function on M; this claim holds, since the factor of automorphy defining the space $\mathcal{S}_1(\Gamma)$ (cf. (2)) is a complex number of modulus one. Second, since all the eigenvalues λ_j are non-negative, it is immediate that $K_M^{(1)}(t; z, z)$ is a monotone decreasing function in t for fixed $z \in M$. Finally, we observe

$$\lim_{t \to \infty} K_M^{(1)}(t; z, z) = \sum_{\lambda_j = 0} |g_{\lambda_j}(z)|^2 = y^2 \sum_{j=1}^{g_M} |f_j(z)|^2 \,,$$

where $\{f_1, ..., f_{g_M}\}$ is an orthonormal basis of $M_2(\Gamma)$ with respect to the Petersson inner product. In particular, we note the formula

$$g_M \cdot \mu_M^{\mathrm{can}}(z) = \lim_{t \to \infty} K_M^{(1)}(t; z, z) \cdot \mu_M^{\mathrm{hyp}}(z).$$

3 The main result

3.1. Theorem. *Let $\Gamma_0 \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a fixed, discrete, cocompact, torsionfree subgroup and $\Gamma \subseteq \Gamma_0$ any subgroup of finite index, i.e., $M = \Gamma \backslash \mathbb{H}$ is a finite cover of the fixed base space $M_0 = \Gamma_0 \backslash \mathbb{H}$. Then, we have the estimate*

$$\mu_M^{\mathrm{can}} = O_{M_0}\left(\mu_M^{\mathrm{shyp}}\right);$$

in other terms, we have

$$\sup_{z \in \mathbb{H}} \sum_{j=1}^{g_M} y^2 |f_j(z)|^2 = O_{M_0}(1).$$

Proof. By 2.4, we have

$$g_M \cdot \mu_M^{\mathrm{can}}(z) = \lim_{t \to \infty} K_M^{(1)}(t; z, z) \cdot \mu_M^{\mathrm{hyp}}(z).$$

Since the function $K_M^{(1)}(t;z,z)$ is monotone decreasing in t for fixed $z \in M$, we obtain the estimate

$$g_M \cdot \mu_M^{\mathrm{can}}(z) \leq K_M^{(1)}(t;z,z) \cdot \mu_M^{\mathrm{hyp}}(z) \leq \sum_{\gamma\in\Gamma} g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,\gamma(z))) \cdot \mu_M^{\mathrm{hyp}}(z)$$

for any $t \geq \delta > 0$. Since $\Gamma_0 \supseteq \Gamma$, this leads to the upper bound

$$g_M \cdot \mu_M^{\mathrm{can}}(z) \leq \mathrm{vol}_M^{\mathrm{hyp}} \cdot \sum_{\gamma\in\Gamma_0} g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,\gamma(z))) \cdot \mu_M^{\mathrm{shyp}}(z).$$

Because of the uniform boundedness of the right-hand side for fixed $t \geq \delta > 0$ and $z \in M_0$, and the relation $\mathrm{vol}_M^{\mathrm{hyp}} = 4\pi(g_M - 1)$, the claim follows. □

References

[1] *Abbes, A.; Ullmo, E.:* Comparaison des métriques d'Arakelov et de Poincaré sur $X_0(N)$. Duke Math. J. 80 (1995), 295-307.

[2] *Chavel, I.:* Eigenvalues in Riemannian Geometry. Pure Appl. Math. **115**. Academic Press, New York, 1984.

[3] *D'Hoker, E.; Phong, D.:* On determinants of Laplacians on Riemann surfaces. Commun. Math. Phys. **104** (1986), 537-545.

[4] *Donnelly, H.:* Elliptic operators and covers of Riemannian manifolds. Math. Z. **223** (1996), 303-308.

[5] *Fay, J.:* Fourier coefficients of the resolvent for a Fuchsian group. J. Reine Angew. Math. **294** (1977), 143-203.

[6] *Hejhal, D.:* The Selberg trace formula for $\mathrm{PSL}_2(\mathbb{R})$, vol. 2. Lecture Notes in Math. **1001**. Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[7] *Rhodes, J.:* Sequences of metrics on Riemann surfaces. Duke Math. J. **72** (1994), 725-738.

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