An integral characterizing the Andreotti-Mayer locus

J. Kramer and R. Salvati Manni

1 Introduction

1.1. Let $\mathcal{A}$ denote the moduli space of n-dimensional, principally polarized, abelian varieties and $p : A \to \mathcal{A}$ the corresponding universal abelian variety. Let $\Theta \subseteq A$, and $p' : \Theta \to \mathcal{A}$ denote the universal theta divisor. For $\tau \in \mathcal{A}$, denote by A_τ, resp. Θ_τ, the fiber of p, resp. p', over τ. Let $\Theta_{\mathrm{sing}} \subseteq \Theta$ be the locus on Θ defined by the union of the singular points of Θ_τ. This can be considered as a subscheme of Θ or A depending on the situation. We set $\Theta^0 := \Theta \setminus \Theta_{\mathrm{sing}}$. The Andreotti-Mayer locus is then defined by

$$\mathcal{N}_0 := \{\tau \in \mathcal{A} \,|\, \Theta_\tau \text{ singular}\}.$$

It can be shown that $\mathcal{N}_0$ is a divisor of $\mathcal{A}$, which has at most two components, and that the equality $\mathcal{N}_0 = p_* \Theta_{\mathrm{sing}}$ holds (cf. [6], [12]). By D. Mumford [12], we know that there exists a Siegel modular form F_n of weight $k = n!(n+3)/2$ with respect to $\mathrm{Sp}_n(\mathbb{Z})$ (and some character χ) satisfying

$$\mathrm{div}(F_n) = \mathcal{N}_0.$$

1.2. In the above cited paper Mumford raised the following question: Is there an explicit polynomial in the theta constants, or other modular forms constructed from theta series (with quadratic forms and pluri-harmonic coefficients) whose zeros give $\mathcal{N}_0$ with suitable multiplicities? An explicit representation of Mumford's modular form F_n is known for $n \leq 4$. In fact, there is an explicit equation for the component of $\mathcal{N}_0$ parametrizing the Θ_τ with a singular point of order 2. The equation describing the other component is rather mysterious (see the end of the introduction of [12]).
In this paper we will give an integral representation of Mumford's modular form F_n, or rather, of $\log \|F_n\|^2_{\mathrm{Pet}}$, where $\|\cdot\|_{\mathrm{Pet}}$ refers to the Petersson metric on the line bundle of Siegel modular forms of weight $k = n!(n+3)/2$, i.e.,

$$\|F_n(\tau)\|^2_{\mathrm{Pet}} = |F_n(\tau)|^2 \cdot \det(\mathrm{Im}\tau)^k.$$

In [14], K.-I. Yoshikawa computed the analytic torsion of Θ_τ, $\tau \in \mathcal{A}$, equipped with the standard translation invariant hermitian metric. In Remark 5.1 of [14], it is mentioned how the formula of our Corollary 4.5 can be obtained by using the formalism of Bismut-Lebeau [1] (adapted to the present singular situation). Our method of proof is different from the one used by K.-I. Yoshikawa. We directly construct the specific Green's currents involved and determine their star-product (for the definitions of Green's currents and star-products, we refer to [13]).
This work is in the spirit of the papers [9] and [10], where now - as new ingredient - not only Green's currents for line bundles, but also for vector bundles of rank bigger than one are considered. The relation to Yoshikawa's computation of the analytic torsion can be seen as a potential arithmetic Riemann-Roch formula for the theta divisor. Since this divisor is singular such a formula is not available yet.

1.3. Concerning our notation we emphasize that the statements made in sections 2 and 3 rather hold true on a smooth finite covering of $\mathcal{A}$ than on $\mathcal{A}$ itself, e.g., on the moduli space associated to the congruence subgroup $\Gamma_n(4) \subseteq \mathrm{Sp}_n(\mathbb{Z})$. But, since the Andreotti-Mayer locus $\mathcal{N}_0$ is $\mathrm{Sp}_n(\mathbb{Z})$-invariant, our main result of Corollary 3.2 obviously descends to hold true on $\mathcal{A}$, which is the reason for this slight abuse of notation.

Acknowledgement. We would like to thank the referee for various helpful remarks which led to a significant improvement of the original version of this paper.

2 Notations and lemmas

2.1. We need the following additional notations: Let L be the line bundle on A associated to Θ. Denote by ϑ the canonical global section of L, i.e.,

$$\vartheta(\tau, z) = \sum_{m \in \mathbb{Z}^n} e^{\pi i m^t \tau m + 2\pi i m^t z}.$$

Equip L with the translation invariant, smooth, hermitian metric $\|\cdot\|_L$ given by

$$\|\vartheta(\tau, z)\|_L^2 := |\vartheta(\tau, z)|^2 \cdot e^{-2\pi (\mathrm{Im} z)^t (\mathrm{Im}\tau)^{-1} (\mathrm{Im} z)} \cdot \det(\mathrm{Im}\tau)^{1/2},$$

and put $\overline{L} := (L, \|\cdot\|_L)$.

Let $T_{A/\mathcal{A}}$, resp. $\Omega^1_{A/\mathcal{A}}$ denote the relative tangent, resp. cotangent bundle of A over $\mathcal{A}$. Equip $T_{A/\mathcal{A}}$ with the smooth, hermitian metric $\|\cdot\|_{T_{A/\mathcal{A}}}$ given by

$$\|X(\tau, z)\|^2_{T_{A/\mathcal{A}}} := X(\tau, z)^t \cdot \mathrm{Im}\tau \cdot \overline{X}(\tau, z)$$

for a vector field $X(\tau,z)$, and put $\overline{T}_{A/\mathcal{A}} := (T_{A/\mathcal{A}}, \|\cdot\|_{T_{A/\mathcal{A}}})$. Equip $\Omega^1_{A/\mathcal{A}}$ with the dual metric $\|\cdot\|_{\Omega^1_{A/\mathcal{A}}} := \|\cdot\|^{-1}_{T_{A/\mathcal{A}}}$, and put $\overline{\Omega}^1_{A/\mathcal{A}} := (\Omega^1_{A/\mathcal{A}}, \|\cdot\|_{\Omega^1_{A/\mathcal{A}}})$.

Equip $\Omega^1_{A/\mathcal{A}} \otimes L$ with the product metric $\|\cdot\|_{\Omega^1_{A/\mathcal{A}}\otimes L} := \|\cdot\|_{\Omega^1_{A/\mathcal{A}}} \cdot \|\cdot\|_L$, i.e., $\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L} = (\Omega^1_{A/\mathcal{A}} \otimes L, \|\cdot\|_{\Omega^1_{A/\mathcal{A}}\otimes L})$.

Furthermore, introduce

$$E := (\Omega^1_{A/\mathcal{A}} \otimes L)|_\Theta \ , \ \|\cdot\|_E := \|\cdot\|_{\Omega^1_{A/\mathcal{A}}\otimes L}|_\Theta,$$

and put $\overline{E} := (E, \|\cdot\|_E)$. We note that $\mathrm{d}\vartheta$ is a global section of E.

2.2. Lemma. *Let g_1 denote an Euler-Green's current for the hermitian line bundle $\overline{L}$ on A associated to the global section ϑ. Then, g_1 satisfies the* $\mathrm{dd^c}$*-equation*

$$\mathrm{dd^c} g_1 + \delta_\Theta = \mathrm{c}_1(\overline{L}),$$

where $\mathrm{c}_1(\overline{L})$ is the first Chern form of $\overline{L}$. On $A \setminus \Theta$, we have

$$g_1|_{A\setminus\Theta}(\tau,z) = -\log\|\vartheta(\tau,z)\|^2_L.$$

Proof. The proof is an immediate consequence of the Poincaré-Lelong lemma. □

2.3. Lemma. *Let g_2 denote an Euler-Green's current for the hermitian vector bundle $\overline{E}$ on the singular variety Θ associated to the global section $\mathrm{d}\vartheta$. Then, g_2 satisfies the* $\mathrm{dd^c}$*-equation*

$$\mathrm{dd^c} g_2 + \delta_{\Theta_{\mathrm{sing}}} = \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})|_\Theta,$$

where $\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})|_\Theta$ is the restriction of the n-th Chern form of $\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L}$ to Θ; here Θ_{sing} is considered as a subscheme of Θ. On $\Theta \setminus \Theta_{\mathrm{sing}} = \Theta^0$, we have

$$g_2|_{\Theta^0}(\tau,z) = -\log\|\mathrm{d}\vartheta(\tau,z)\|^2_E \wedge \sum_{j=0}^{n-1} \zeta_1^j \wedge \mathrm{c}_{n-1-j}(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})|_{\Theta^0} + \zeta_2,$$

where ζ_1 is a $(1,1)$-form and ζ_2 a $(n-1,n-1)$-form on Θ^0, which will be made explicit in course of the proof of Lemma 2.3.

Proof. Our proof is based on a construction given in [4], section 1.2.3, pp. 915-916. In order to be able to follow [4], we additionally have to use a desingularization.

From [5] we recall that Θ_{sing} is a local complete intersection, since its irreducible components have codimension $n+1$ in A. We denote by Θ' the blow-up of Θ

in $\Theta_{\rm sing}$, by $\pi' : \Theta' \longrightarrow \Theta$ the natural projection, and by D' the exceptional divisor.

Consider now the global section $\mathrm{d}\vartheta$ of E; it gives rise to a morphism

$$\varphi^0 : \Theta^0 \longrightarrow \mathbb{P}(E).$$

This morphism extends to a closed immersion

$$\varphi' : \Theta' \longrightarrow \mathbb{P}(E).$$

Let $\tilde{\Theta}$ be a desingularization of Θ' à la Hironaka. Then, the closed immersion φ' lifts to a morphism $\varphi : \tilde{\Theta} \longrightarrow \mathbb{P}(E)$. Denoting by $\tilde{\pi} : \tilde{\Theta} \longrightarrow \Theta'$ the natural projection, and setting $\tilde{D} := \tilde{\pi}^* D'$, we obtain the following commutative diagram:

$$\begin{tikzcd}
\tilde{D} \arrow[r] \arrow[d] & \tilde{\Theta} \arrow[d, "\tilde{\pi}"] \arrow[dr, "\varphi"] & & \\
D' \arrow[r] \arrow[d] & \Theta' \arrow[r, "\varphi'"] \arrow[d, "\pi'"] & \mathbb{P}(E) \arrow[r, "\iota'"] \arrow[dl, "\psi'"'] & \mathbb{P}(\Omega^1_{A/\mathcal{A}} \otimes L) \arrow[dl, "\psi"] \\
\Theta_{\rm sing} \arrow[r] & \Theta \arrow[r, "\iota"] & A &
\end{tikzcd}$$

Here ψ, ψ' are the natural projections, and ι, ι' are the natural embeddings.

On $\mathbb{P}(\Omega^1_{A/\mathcal{A}} \otimes L)$ we have the tautological short exact sequence of vector bundles

$$\mathcal{E}' : \ 0 \longrightarrow S'^{\vee} \longrightarrow \psi^*(\Omega^1_{A/\mathcal{A}} \otimes L) \longrightarrow Q' \longrightarrow 0,$$

where S', resp. Q' is the tautological line bundle $\mathcal{O}_{\mathbb{P}(\Omega^1_{A/\mathcal{A}} \otimes L)}(1)$, resp. the tautological quotient vector bundle of rank $(n-1)$. By means of the short exact sequence $\mathcal{E}'$, the hermitian metric on $\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L}$ induces hermitian metrics $\|\cdot\|_{S'}$ on S', resp. $\|\cdot\|_{Q'}$ on Q'; we set $\overline{S}' := (S', \|\cdot\|_{S'})$, resp. $\overline{Q}' := (Q', \|\cdot\|_{Q'})$, and denote by $\overline{\mathcal{E}}'$ the short exact sequence $\mathcal{E}'$ equipped with the metrics described above. We note that $\|\cdot\|_{S'}$ is nothing but the Fubini-Study metric on S'. From the Whitney formula and by functoriality, we deduce

$$\begin{aligned} &\mathrm{c}_n\left(\overline{S}' \otimes \psi^*(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})\right) - \\ &\sum_{j=0}^{n} \mathrm{c}_1(\overline{S}')^j \wedge \psi^* \mathrm{c}_{n-j}(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L}) = \mathrm{dd^c}\eta', \end{aligned} \tag{1}$$

where the $(n-1,n-1)$-form η' is given by the Bott-Chern secondary form

$$\eta' = \widetilde{\mathrm{c}}_n(\overline{\mathcal{E}}' \otimes \overline{S}')$$

associated to the n-th Chern form $\mathrm{c}_n(\cdot)$; it is unique up to $\operatorname{im}\partial + \operatorname{im}\overline{\partial}$ (cf. [7]). With

$$\overline{S} := (\iota' \circ \varphi)^* \overline{S}' \,,\ \overline{Q} := (\iota' \circ \varphi)^* \overline{Q}',$$

we obtain the short exact sequence of vector bundles on $\tilde{\Theta}$

$$\mathcal{E}:\ 0 \longrightarrow S^\vee \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0,$$

where $\pi := \pi' \circ \tilde{\pi}$. Since S is the pull-back (via $\iota' \circ \varphi$ to $\tilde{\Theta}$) of the tautological line bundle S' on $\mathbb{P}(\Omega^1_{A/\mathcal{A}} \otimes L)$, our construction shows

$$S^\vee = \mathcal{O}_{\tilde{\Theta}}(\tilde{D}).$$

The Green's current associated to the canonical section $\sigma = \pi^* \mathrm{d}\vartheta$ of $S^\vee = \mathcal{O}_{\tilde{\Theta}}(\tilde{D})$ therefore satisfies the $\mathrm{dd^c}$-equation

$$\mathrm{dd^c}\left(-\log\|\sigma\|^2_{S^\vee}\right) + \delta_{\tilde{D}} = \mathrm{c}_1(\overline{S}^\vee) = -\mathrm{c}_1(\overline{S}). \tag{2}$$

Multiplying (2) with the $(n-1,n-1)$-form α given by the formula

$$\begin{aligned}
\alpha \quad &:= \quad (\iota' \circ \varphi)^* \left(\sum_{j=0}^{n-1} \mathrm{c}_1\left(\overline{S}'\right)^j \wedge \mathrm{c}_{n-1-j}\left(\psi^*(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})\right)\right) \\
&= \quad \varphi^* \left(\sum_{j=0}^{n-1} \mathrm{c}_1\left(\iota'^*\,\overline{S}'\right)^j \wedge \mathrm{c}_{n-1-j}\left(\psi'^*\overline{E}\right)\right) \\
&= \quad \sum_{j=0}^{n-1} \mathrm{c}_1\left(\overline{S}\right)^j \wedge \mathrm{c}_{n-1-j}\left(\pi^*\overline{E}\right),
\end{aligned}$$

we derive from (1)

$$\begin{aligned}
&\mathrm{dd^c}\left(-\log\|\sigma\|^2_{S^\vee} \wedge \alpha\right) + \delta_{\tilde{D}} \wedge \alpha = -\mathrm{c}_1(\overline{S}) \wedge \alpha = \\
&\mathrm{c}_n(\pi^*\overline{E}) - \mathrm{dd^c}\eta,
\end{aligned}$$

where η is given by the functoriality of Bott-Chern secondary forms by

$$\eta = \widetilde{\mathrm{c}}_n(\overline{\mathcal{E}} \otimes \overline{S}).$$

A standard cohomological argument shows

$$\pi_*\left(\delta_{\tilde{D}} \wedge \alpha\right) = \delta_{\Theta_{\mathrm{sing}}}.$$

Putting

$$g_2 := \pi_*\left(-\log\|\pi^*\mathrm{d}\vartheta\|^2_{S^\vee}\wedge\alpha+\eta\right),$$

we get after a short calculation

$$\mathrm{dd^c}g_2+\delta_{\Theta_{\mathrm{sing}}}=\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})|_\Theta.$$

Since $\pi:\tilde{\Theta}\setminus\tilde{D}\longrightarrow\Theta^0$ is an isomorphism, the definition of g_2 shows the explicit formula

$$g_2|_{\Theta^0}(\tau,z)=-\log\|\mathrm{d}\vartheta(\tau,z)\|^2_E\wedge\sum_{j=0}^{n-1}\zeta_1^j\wedge\mathrm{c}_{n-1-j}(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})|_{\Theta^0}+\zeta_2,$$

with the $(1,1)$-form $\zeta_1:=\pi_*(\mathrm{c}_1(\overline{S})|_{\tilde{\Theta}\setminus\tilde{D}})$, and the $(n-1,n-1)$-form $\zeta_2:=\pi_*(\eta)$ on Θ^0. This proves the lemma. □

2.4. Remark. With the above notations, let $\overline{H}:=(H,\|\cdot\|_{\mathrm{FS}})$ denote the tautological line bundle $H=\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ equipped with the Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$, i.e., $\overline{H}=\iota'^*\overline{S}'$. The isomorphism $\pi:\tilde{\Theta}\setminus\tilde{D}\longrightarrow\Theta^0$ (cf. proof of Lemma 2.3) then leads to the isomorphism of hermitian line bundles

$$(\varphi^0)^*\overline{H}\cong\overline{S}|_{\tilde{\Theta}\setminus\tilde{D}}.$$

3 The main result

3.1. Theorem. *With the above notations, define the following current of type (n,n) on A*

$$g:=g_1\wedge\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})+g_2\wedge\delta_\Theta.$$

Then, g is a Green's current on A satisfying

$$\mathrm{dd^c}g+\delta_{\Theta_{\mathrm{sing}}}=\mathrm{c}_1(\overline{L})\wedge\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L});\tag{3}$$

here Θ_{sing} is considered as a subscheme of A.

Proof. A straightforward computation yields

$$\begin{aligned}\mathrm{dd^c}g &= \mathrm{dd^c}(g_1)\wedge\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})+\mathrm{dd^c}(g_2)\wedge\delta_\Theta\\ &= \left(\mathrm{c}_1(\overline{L})-\delta_\Theta\right)\wedge\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})\\ &+ \left(\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})|_\Theta-\delta_{\Theta_{\mathrm{sing}}}\right)\wedge\delta_\Theta\\ &= \mathrm{c}_1(\overline{L})\wedge\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})-\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})|_\Theta\\ &+ \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})|_\Theta-\delta_{\Theta_{\mathrm{sing}}}\wedge\delta_\Theta\\ &= \mathrm{c}_1(\overline{L})\wedge\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}}\otimes\overline{L})-\delta_{\Theta_{\mathrm{sing}}},\end{aligned}$$

as claimed. □

3.2. Corollary. *With the above notations, consider the current* p_*g *of type* $(0,0)$ *on* $\mathcal{A}$. *Then,* p_*g *is a Green's function on* $\mathcal{A}$ *satisfying*

$$\mathrm{dd^c}(p_*g) + \delta_{\mathcal{N}_0} = \mathrm{c}_1(\overline{\omega}^{\otimes k}), \tag{4}$$

where $\overline{\omega} = (\omega, \|\cdot\|_{\mathrm{Pet}})$ *with* $\omega = \det p_*\Omega^1_{A/\mathcal{A}}$ *the canonical line bundle on* $\mathcal{A}$ *equipped with the Petersson metric* $\|\cdot\|_{\mathrm{Pet}}$, *and*

$$k = \frac{n!\,(n+3)}{2}\,.$$

Proof. Applying p_* to the $\mathrm{dd^c}$-equation (3) given in Theorem 3.1 leads to the equation

$$\mathrm{dd^c}(p_*g) + \delta_{p_*\Theta_{\mathrm{sing}}} = p_*\left(\mathrm{c}_1(\overline{L}) \wedge \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})\right).$$

From 1.1, we recall $p_*\Theta_{\mathrm{sing}} = \mathcal{N}_0$. In order to simplify the right hand side of the above equation, we first note that $\overline{\Omega}^1_{A/\mathcal{A}} = p^*\overline{F}$ with a suitable hermitian vector bundle $\overline{F}$ of rank n on $\mathcal{A}$ satisfying $\det\overline{F} = \overline{\omega}$. With this in mind, we compute

$$p_*\left(\mathrm{c}_1(\overline{L}) \wedge \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})\right) = p_*\left(\mathrm{c}_1(\overline{L}) \wedge \sum_{j=0}^{n} \mathrm{c}_j(p^*\overline{F}) \wedge \mathrm{c}_1(\overline{L})^{n-j}\right) =$$
$$\sum_{j=0}^{n} \mathrm{c}_j(\overline{F}) \wedge p_*\left(\mathrm{c}_1(\overline{L})^{n+1-j}\right).$$

From [9], Remark 5.3, we know

$$p_*(\mathrm{c}_1(\overline{L})^{n+1}) = \frac{(n+1)!}{2} \cdot \mathrm{c}_1(\overline{\omega});$$

furthermore, we have $p_*(\mathrm{c}_1(\overline{L})^n) = n!$. Since $p_*(\mathrm{c}_1(\overline{L})^{n+1-j}) = 0$ for dimension reasons, if $j \geq 2$, we finally obtain

$$p_*\left(\mathrm{c}_1(\overline{L}) \wedge \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})\right) =$$
$$\frac{(n+1)!}{2} \cdot \mathrm{c}_1(\overline{\omega}) + n! \cdot \mathrm{c}_1(\det\overline{F}) = \frac{n!\,(n+3)}{2} \cdot \mathrm{c}_1(\overline{\omega}).$$

This proves the claim. □

3.3. Remark. The proof of Corollary 3.2 could also be completed as follows. From [12], pp. 372-373, we have on the level of Chern classes

$$p_*\left(\mathrm{c}_1(L) \cdot \mathrm{c}_n(\Omega^1_{A/\mathcal{A}} \otimes L)\right) = \mathrm{c}_1(\omega^{\otimes k}) \tag{5}$$

with
$$k = \frac{n!\,(n+3)}{2}\,.$$
We are left to show that equality (5) continues to hold true on the level of Chern forms. To do this, we consider the exact $(1,1)$-from
$$\zeta := p_* \left(\mathrm{c}_1(\overline{L}) \wedge \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L}) \right) - \mathrm{c}_1(\overline{\omega}^{\otimes k})$$
on $\mathcal{A}$, which will be shown to vanish. For any positive integer N, let $\Gamma_n(N)$ denote the principal congruence subgroup of $\mathrm{Sp}_n(\mathbb{Z})$ of level N. Whenever N' divides N, let
$$p_{N,N'} : \Gamma_n(N)\backslash\mathbb{H}_n \longrightarrow \Gamma_n(N')\backslash\mathbb{H}_n$$
denote the natural projection. The set of modular varieties $\Gamma_n(N)\backslash\mathbb{H}_n$ together with the projections $p_{N,N'}$ $(N'|N)$ gives rise to a projective system on which the group $\mathrm{Sp}_n(\mathbb{Q})$ acts. Therefore, $\mathrm{Sp}_n(\mathbb{Q})$ also acts on the corresponding inverse limit
$$\mathcal{H}_n := \varprojlim_N \Gamma_n(N)\backslash\mathbb{H}_n.$$
The pull-back of ζ to $\mathcal{H}_n$ is $\mathrm{Sp}_n(\mathbb{Q})$-invariant, and hence the pull-back of ζ to the universal cover $\mathbb{H}_n$ is also $\mathrm{Sp}_n(\mathbb{Q})$-invariant. The denseness of $\mathrm{Sp}_n(\mathbb{Q})$ in $\mathrm{Sp}_n(\mathbb{R})$ now shows that the pull-back of ζ to $\mathbb{H}_n$ is $\mathrm{Sp}_n(\mathbb{R})$-invariant, and hence harmonic on $\mathbb{H}_n$ (cf. [2], p. 241, or [8], chapter II.7, in particular the proof of theorem 7.8). Therefore, ζ is harmonic on $\mathcal{A}$.
Furthermore, the differential form ζ is seen to be good in the sense of [11], and therefore extends as a current $[\zeta]$ to a smooth toroidal compactification $\hat{\mathcal{A}}$ of $\mathcal{A}$. For any smooth form φ on $\hat{\mathcal{A}}$, we then obtain after applying Stokes' theorem
$$\Delta[\zeta](\varphi) = \int_{\hat{\mathcal{A}}} \zeta \wedge \Delta\varphi = \int_{\hat{\mathcal{A}}} \Delta\zeta \wedge \varphi = \int_{\mathcal{A}} \Delta\zeta \wedge \varphi = 0$$
by the harmonicity of ζ on $\mathcal{A}$. This shows that $[\zeta]$ is a harmonic current on $\hat{\mathcal{A}}$. Since ζ is exact, another application of Stokes' theorem shows that $[\zeta] \in \mathrm{im}\,\mathrm{d}$. The Hodge decomposition for currents on $\hat{\mathcal{A}}$ implies $\zeta = 0$ on $\mathcal{A}$. This proves the claim.
We note that we could also have concluded from Theorem 4.4 of [3] that ζ vanishes as long as $n \geq 3$.

3.4. Remark. Mumford's Siegel modular form F_n of weight $k = n!(n+3)/2$ mentioned in 1.1 satisfies the $\mathrm{dd^c}$-equation
$$\mathrm{dd^c}(-\log \|F_n\|^2_{\mathrm{Pet}}) + \delta_{\mathcal{N}_0} = \mathrm{c}_1(\overline{\omega}^{\otimes k}). \tag{6}$$
Subtracting (6) from (4) then yields
$$\mathrm{dd^c}(p_* g + \log \|F_n\|^2_{\mathrm{Pet}}) = 0$$

on $\mathcal{A}$, i.e., $p_*g + \log \|F_n\|^2_{\mathrm{Pet}}$ is a harmonic function on $\mathcal{A}$. If $n \geq 2$, the Koecher principle shows that this function has to be constant, i.e.,

$$p_*g = -\log \|F_n\|^2_{\mathrm{Pet}} + c_n$$

for a suitable real constant c_n. Putting

$$\Delta_n := e^{-c_n/2} \cdot F_n,$$

we find a Siegel modular form of weight k with respect to $\mathrm{Sp}_n(\mathbb{Z})$ (and some character χ) vanishing precisely along the Andreotti-Mayer locus $\mathcal{N}_0$, and satisfying

$$p_*g = -\log \|\Delta_n\|^2_{\mathrm{Pet}}.$$

For $n = 1$ the same result follows by additionally taking into account that the function $p_*g + \log \|F_n\|^2_{\mathrm{Pet}}$ is bounded at the cusp at infinity, which follows from the explicit description of p_*g given in the next section.

4 The integral representation

4.1. By means of Theorem 3.1 and Corollary 3.2 we are now able to give an integral representation for the Siegel modular form $\Delta_n(\tau)$, or rather, for $\log \|\Delta_n(\tau)\|^2_{\mathrm{Pet}}$ for $\tau \in \mathcal{A} \setminus \mathcal{N}_0$.
From the definition of the Green's current g given in Theorem 3.1 we derive for $\tau \in \mathcal{A} \setminus \mathcal{N}_0$:

$$\begin{aligned}
(p_*g)(\tau) \quad &= \quad p_*\Big(g_1 \wedge \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L}) + g_2 \wedge \delta_\Theta\Big)(\tau) = \\
&- \int\limits_{A_\tau} \log \|\vartheta(\tau,z)\|^2_L \wedge \mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})|_{A_\tau} \\
&- \int\limits_{\Theta_\tau} \log \|\mathrm{d}\vartheta(\tau,z)\|^2_E \wedge \sum_{j=0}^{n-1} (\zeta_1|_{\Theta_\tau})^j \wedge \mathrm{c}_{n-1-j}(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})|_{\Theta_\tau} \\
&+ \int\limits_{\Theta_\tau} (\zeta_2|_{\Theta_\tau}).
\end{aligned}$$

We denote by $\overline{L}_{A_\tau}$, resp. $\overline{L}_{\Theta_\tau}$ the restriction of $\overline{L}$ to A_τ, resp. Θ_τ, and by $\overline{\Omega}^1_{A_\tau}$, resp. $\overline{\Omega}^1_{\Theta_\tau}$ the restriction of $\overline{\Omega}^1_{A/\mathcal{A}}$ to A_τ, resp. Θ_τ (note that $\tau \in \mathcal{A} \setminus \mathcal{N}_0$). Since $\overline{\Omega}^1_{A_\tau}$ is trivial, we have by functoriality

$$\mathrm{c}_n(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})|_{A_\tau} = \mathrm{c}_n(\overline{\Omega}^1_{A_\tau} \otimes \overline{L}_{A_\tau}) = \mathrm{c}_1(\overline{L}_{A_\tau})^n.$$

For the same reason, we derive for $j = 0, ..., n-1$

$$\mathrm{c}_{n-1-j}(\overline{\Omega}^1_{A/\mathcal{A}} \otimes \overline{L})|_{\Theta_\tau} = \mathrm{c}_{n-1-j}(\overline{\Omega}^1_{\Theta_\tau} \otimes \overline{L}_{\Theta_\tau}) = \mathrm{c}_1(\overline{L}_{\Theta_\tau})^{n-1-j}.$$

4.2. From Lemma 2.3 and Remark 2.4 we recall the morphism

$$\varphi^0 : \Theta^0 \longrightarrow \mathbb{P}(E)$$

together with the tautological line bundle $\overline{H} = (\mathcal{O}_{\mathbb{P}(E)}(1), \|\cdot\|_{\mathrm{FS}})$ equipped with the Fubini-Study metric. Restricting φ^0 to the non-singular fiber Θ_τ (note that $\tau \in \mathcal{A} \setminus \mathcal{N}_0$), we obtain the morphism

$$\varphi_\tau : \Theta_\tau \longrightarrow \mathbb{P}(\Omega^1_{\Theta_\tau} \otimes L_{\Theta_\tau})$$

together with the tautological line bundle $\overline{H}_{\Theta_\tau} := \overline{\mathcal{O}}_{\mathbb{P}(\Omega^1_{\Theta_\tau} \otimes L_{\Theta_\tau})}(1)$. Identifying $\tilde{\Theta}_\tau$ (the fiber of $\tilde{\Theta}$ lying over τ) with Θ_τ, we can put $\overline{S}_{\Theta_\tau} := \overline{S}|_{\Theta_\tau}$ with the tautological line bundle $\overline{S}$ on $\tilde{\Theta}$ (cf. proof of Lemma 2.3). With this identification we derive from Remark 2.4 the equality

$$\varphi_\tau^* \overline{H}_{\Theta_\tau} = \overline{S}_{\Theta_\tau}$$

which leads to

$$\zeta_1|_{\Theta_\tau} = \mathrm{c}_1(\overline{S})|_{\Theta_\tau} = \mathrm{c}_1(\varphi_\tau^* \overline{H}_{\Theta_\tau}).$$

4.3. Since $\Omega^1_{\Theta_\tau}$ is trivial, the projective bundle $\mathbb{P}(\Omega^1_{\Theta_\tau} \otimes L_{\Theta_\tau})$ is trivial, and we get an isomorphism

$$\beta_\tau : \mathbb{P}(\Omega^1_{\Theta_\tau} \otimes L_{\Theta_\tau}) = \mathbb{P}(L_{\Theta_\tau} \oplus ... \oplus L_{\Theta_\tau}) \longrightarrow \Theta_\tau \times \mathbb{P}^{n-1}.$$

We now denote by $\mathrm{pr}_{1,\tau}$, resp. $\mathrm{pr}_{2,\tau}$ the projection from $\Theta_\tau \times \mathbb{P}^{n-1}$ to the first, resp. second factor, and set $\alpha_\tau = \mathrm{pr}_{2,\tau} \circ \beta_\tau \circ \varphi_\tau$; we note that $\mathrm{pr}_{1,\tau} \circ \beta_\tau \circ \varphi_\tau = \mathrm{id}$. Furthermore, we denote by $\overline{\mathcal{E}}_{\Theta_\tau}$ the restriction of the short exact sequence $\overline{\mathcal{E}}$ to $\tilde{\Theta}_\tau \cong \Theta_\tau$. We denote by $\overline{\mathcal{E}}_{\mathbb{P}^{n-1}}$ the tautological short exact sequence on $\mathbb{P}^{n-1}$ equipped with the hermitian metrics induced by the hermitian metric given by

$$\|w\|^2 = w^t \cdot (\mathrm{Im}\tau)^{-1} \cdot w$$

on the trivial rank-n bundle on $\mathbb{P}^{n-1}$. By our constructions, we have

$$\overline{\mathcal{E}}_{\Theta_\tau} = \alpha_\tau^*(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}}) \otimes \overline{L}_{\Theta_\tau}\,,\ \overline{S}_{\Theta_\tau} = \alpha_\tau^*(\overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1)) \otimes \overline{L}^{\vee}_{\Theta_\tau},$$

which leads to

$$\overline{\mathcal{E}}_{\Theta_\tau} \otimes \overline{S}_{\Theta_\tau} = \alpha_\tau^*(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}} \otimes \overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1)).$$

By the functoriality of Bott-Chern secondary classes, we find

$$\zeta_2|_{\Theta_\tau} = \pi_*(\eta)|_{\Theta_\tau} = \widetilde{\mathrm{c}}_n(\overline{\mathcal{E}}_{\Theta_\tau} \otimes \overline{S}_{\Theta_\tau}) = \alpha_\tau^* \,\widetilde{\mathrm{c}}_n(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}} \otimes \overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1)).$$

Keeping in mind that for fixed $\tau \in \mathcal{A} \setminus \mathcal{N}_0$, the morphism α_τ is finite and has degree $n!$, we compute

$$\int_{\Theta_\tau} \zeta_2|_{\Theta_\tau} = \int_{\Theta_\tau} \alpha_\tau^* \widetilde{\mathrm{c}}_n(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}} \otimes \overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1)) =$$

$$n! \int_{\mathbb{P}^{n-1}} \widetilde{\mathrm{c}}_n(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}} \otimes \overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1)) =$$

$$n! \sum_{j=0}^{n-1} \int_{\mathbb{P}^{n-1}} \widetilde{\mathrm{c}}_{j+1}(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}}) \cdot \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1))^{n-1-j}. \tag{7}$$

Since we recognize by an obvious change of variables that the differential forms

$$\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1)),\ \widetilde{\mathrm{c}}_{j+1}(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}}) \quad (j = 0, ..., n-1)$$

are pull-backs of the corresponding differential forms on $\mathbb{P}^{n-1}$ associated to the tautological short exact sequence equipped with the standard hermitian metrics which are independent of τ, we find that the integral

$$\int_{\Theta_\tau} \zeta_2|_{\Theta_\tau}$$

is independent of τ, and hence equal to a constant C_n.

4.4. In fact, the constant C_n can be explicitly computed. Namely, using results from [7], pp. 209-212, we derive from formula (7)

$$\begin{aligned} C_n &= n! \sum_{j=0}^{n-1} \int_{\mathbb{P}^{n-1}} \widetilde{\mathrm{c}}_{j+1}(\overline{\mathcal{E}}_{\mathbb{P}^{n-1}}) \cdot \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1))^{n-1-j} \\ &= -n! \left(\sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{1}{k} \right) \int_{\mathbb{P}^{n-1}} \mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1))^{n-1} \\ &= -n! \left(\sum_{j=1}^{n-1} \sum_{k=1}^{j} \frac{1}{k} \right). \end{aligned}$$

We note that, up to the factor $-n!$, the constant C_n is equal to the arithmetic degree of the hermitian line bundle $\overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1)$.

Collecting the above results then leads to the following corollary:

4.5. Corollary. *The Siegel modular form Δ_n from Remark 3.4 is characterized* *by the following integral representation*

$$\log \|\Delta_n(\tau)\|^2_{\mathrm{Pet}} = \int_{A_\tau} \log \|\vartheta(\tau,z)\|^2_L \wedge \mathrm{c}_1(\overline{L}_{A_\tau})^n +$$

$$\int_{\Theta_\tau} \log \|\mathrm{d}\vartheta(\tau,z)\|^2_E \wedge \sum_{j=0}^{n-1} \mathrm{c}_1(\varphi_\tau^* \overline{H}_{\Theta_\tau})^j \wedge \mathrm{c}_1(\overline{L}_{\Theta_\tau})^{n-1-j} + C_n$$

with the constant C_n given in 4.4. □

4.6. Remark. We now see that the integral representation given in Corollary 4.5 coincides with the one given in [14], Remark 5.1, by taking into account that the left-hand side of Yoshikawa's formula contains the absolute value while our formula contains the Petersson norm of the modular form $\Delta_n(\tau)$, and Yoshikawa's hermitian metric on L differs from ours by a factor $\det(\mathrm{Im}\tau)^{1/2}$. However, we note that our formula contains the first Chern form $\mathrm{c}_1(\varphi_\tau^*\overline{H}_{\Theta_\tau})$ while Yoshikawa's contains, in our notation, $\alpha_\tau^*\mathrm{c}_1(\overline{\mathcal{O}}_{\mathbb{P}^{n-1}}(1))$, which differs from ours by a twist with $\overline{L}^\vee_{\Theta_\tau}$; this difference is explained by the careful identification of the tautological line bundles on $\mathbb{P}(\Omega^1_{\Theta_\tau} \otimes L_{\Theta_\tau})$ and on $\Theta_\tau \times \mathbb{P}^{n-1}$.

References

[1] *Bismut, J.-M.; Lebeau, G.:* Complex immersions and Quillen metrics. Inst. Hautes Études Sci. Publ. Math. **74**, 1991.

[2] *Borel, A.:* Stable real cohomology of arithmetic groups. Ann. Scient. Éc. Norm. Sup. **7** (1974), 235-272.

[3] *Borel, A.:* Stable real cohomology of arithmetic groups II. Progress in Math. **14** (1981), 21-55.

[4] *Bost, J.-B.; Gillet, H.; Soulé, C.:* Heights of projective varieties and positive Green forms. J. Am. Math. Soc. **7** (1994), 903-1027.

[5] *Ciliberto, C.; van der Geer, G.:* The moduli space of abelian varieties and the singularities of the theta divisor. In: Surveys in Differential Geometry: Papers dedicated to Atiyah, Bott, Hirzebruch, Singer. Y.S. Yau (ed.). International Press, 2000, 61-82.

[6] *Debarre, O.:* Le lieu des variétés abéliennes dont le diviseur theta est singulier a deux composantes. Ann. Scient. Ec. Norm. Sup. **25** (1992), 687-708.

[7] *Gillet, H.; Soulé, C.:* Characteristic classes for algebraic vector bundles with hermitian metrics I, II. Annals of Math. **131** (1990), 163-203, 205-238.

[8] *Helgason, S.:* Differential geometry, Lie groups, and symmetric spaces. Academic Press, London-San Diego, 1978.

[9] *Jorgenson, J.; Kramer, J.:* Towards the arithmetic degree of line bundles on abelian varieties. Manuscripta Math. **96** (1998), 335-370.

[10] *Jorgenson, J.; Kramer, J.:* Star products of Green's currents and automorphic forms. Duke Math. J. **106** (2001), 553-580.

[11] *Mumford, D.:* Hirzebruch's proportionality theorem in the non-compact case. Invent. Math. **42** (1977), 239-272.

[12] *Mumford, D.:* On the Kodaira dimension of the Siegel modular variety. Lecture Notes in Math. **997** (1983), 348-375.

[13] *Soulé, C.; Abramovich, D.; Burnol, J.-F.; Kramer, J.:* Lectures on Arakelov geometry. Cambridge Studies in Advanced Mathematics **33**. Cambridge University Press, Cambridge, 1992.

[14] *Yoshikawa, K.-I.:* Discriminant of theta divisors and Quillen metrics. J. Diff. Geom. **52** (1999), 73-115.

Jürg Kramer
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin
Germany
e-mail: kramer@mathematik.hu-berlin.de

Riccardo Salvati Manni
Dipartimento di Matematica
Università di Roma La Sapienza
Piazzale Aldo Moro 2
I-00185 Roma
Italy
e-mail: salvati@mat.uniroma1.it