Bounds for special values of Selberg zeta functions of Riemann surfaces

J. Jorgenson and J. Kramer *

1 Introduction

1.1. In this paper we investigate the following problem: Letting $\mathbb{H}$ denote the upper half plane, $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ any Fuchsian group of the first kind, $M := \Gamma\backslash\mathbb{H}$ the corresponding quotient space and $Z_M(s)$ the Selberg zeta function associated to M, we will give upper and lower bounds for the quantity

$$c_M := \lim_{s\to 1}\left(\frac{Z_M'}{Z_M}(s) - \frac{1}{s-1}\right).$$

The bounds will be given in terms of topological and spectral theoretical invariants of M. The main tool to achieve our results is the hyperbolic heat trace associated to M as defined in [15]. We use the spectral theoretical realization of the hyperbolic heat trace, results from the proof of the Selberg trace formula and methods from analytical number theory to establish our bounds.

More precisely, we prove the following main results: In Theorem 3.3 we establish a lower bound for c_M involving only the volume, the number of cusps and the total branching order of M. In Theorem 4.7 we give an upper bound for c_M in terms of small eigenvalues and short, primitive geodesics of M; applying this result to arithmetic subgroups Γ leads to an upper bound of c_M in terms of short, primitive geodesics only. To state the last main result, we assume that Γ is a subgroup of finite index in a fixed Fuchsian group $\Gamma_0 \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind, i.e., M is a finite cover of $M_0 = \Gamma_0\backslash\mathbb{H}$. Then, we prove in Theorem 4.11 that the quantity c_M can be bounded in terms of the volume of M and small

*The first author acknowledges support from NSF grant DMS-96-22535, from A.G. Petrello and from A. Typaldos and thanks M. Burger and A.-S. Sznitman for the invitation to the FIM at the ETH in Zurich. The second author would like to express his thanks to V.K. Murty for the invitation to the University of Toronto and to J. Oesterlé for the invitation to the Institut Henri Poincaré (Centre Emile Borel - UMS 839 IHP (CNRS/UPMC) - Paris), where parts of this note have been written.

eigenvalues of M only; again, if Γ is arithmetic, the contribution of the small eigenvalues can be eliminated.

Applying our results to $\Gamma_0 = \mathrm{PSL}_2(\mathbb{Z})$ and the congruence subgroup $\Gamma = \Gamma_0(N)$ for square-free N, we are able to derive the estimate

$$c_{\Gamma_0(N)\backslash\mathbb{H}} = O_\varepsilon(N^\varepsilon)$$

for any given $\varepsilon > 0$. The proof of this bound also relies on results of M. Huxley on the number of conjugacy classes of matrices of bounded trace in congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.

1.2. Let us point out now how the quantity c_M is related to Arakelov Geometry: Letting $\mathcal{X}_0(N)/\mathbb{Z}$ denote the minimal regular model of the modular curve $X_0(N)/\mathbb{Q}$ associated to the congruence subgroup $\Gamma_0(N)$ and $\overline{\omega}$ the relative dualizing sheaf of $\mathcal{X}_0(N)$ equipped with the Arakelov metric, A. Abbes and E. Ullmo derived in [1] the following upper bound for the arithmetic self-intersection number $\overline{\omega}^2$ as N tends to infinity, assuming N to be square-free and $2,3 \nmid N$:

$$\begin{aligned} \overline{\omega}^2 \quad &\leq \quad -8\pi \frac{g(X_0(N))-1}{\mathrm{vol}(X_0(N))} \cdot c_{\Gamma_0(N)\backslash\mathbb{H}} + 2g(X_0(N))\log N \\ &+ \quad g(X_0(N)) \sum_{p|N} \frac{p+1}{p-1} \log p + o(g(X_0(N))\log N); \end{aligned}$$

here $g(X_0(N))$, resp. $\mathrm{vol}(X_0(N))$ denotes the genus, resp. the hyperbolic volume of the modular curve $X_0(N)$. In their study of rational points of small height on $X_0(N)$ in [20], P. Michel and E. Ullmo were able to establish as a by-product the estimate $c_{\Gamma_0(N)\backslash\mathbb{H}} = O_\varepsilon(N^{7/8+\varepsilon})$ for any $\varepsilon > 0$, which differs from ours by the additional exponent $7/8$; in any case, both estimates lead to

$$\overline{\omega}^2 = O(g(X_0(N))\log N).$$

Replacing $\overline{\omega}$ by the line bundle of modular forms $\overline{\mathcal{M}}_k$ of weight k equipped with the Petersson metric, which is logarithmically singular at the cusps of $X_0(N)$, U. Kühn in his PhD thesis was able to extend 1-dimensional Arakelov Geometry to such singular metrics and to show that the (generalized) arithmetic self-intersection number $\overline{\mathcal{M}}_k^2$ is given by the formula

$$\overline{\mathcal{M}}_k^2 = k^2 \cdot [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma_0(N)] \cdot \left(\frac{\zeta_{\mathbb{Q}}(-1)}{2} + \zeta'_{\mathbb{Q}}(-1) \right),$$

where $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function (see [16], [17]). Since the line bundles $\omega^{\otimes k/2}$ and $\mathcal{M}_k$ are very closely related to each other, a comparison of the two arithmetic self-intersection numbers suggests that the quantity $c_{\Gamma_0(N)\backslash\mathbb{H}}$ plays the role of a secondary class, which measures the change between the

Arakelov and the Petersson metric. This aspect of the quantity c_M and similar refined invariants of M deserves further study. As the preprint [12] shows, the techniques developed in this paper are useful in treating related arithmetic questions.

1.3. The paper is organized as follows: In section 2, after fixing some notation, we recall the definitions of Selberg zeta functions, of heat kernels and of hyperbolic heat traces, as studied in [15], attached to M. We then recall from [15], p. 656, that the logarithmic derivative of the Selberg zeta function can be realized as a Gauss transform of the hyperbolic heat trace; if M is compact, this formula is sometimes referred to as McKean's formula, where the hyperbolic heat trace in this case is the trace of the heat kernel minus the volume of M times the heat kernel of the upper half plane $\mathbb{H}$ evaluated at distance zero. After recalling the definition of the prime geodesic counting function, we conclude this section by reminding the reader of the statement of the prime geodesic theorem.

In section 3, we establish the lower bound for c_M (see Theorem 3.3). For its proof we start from the representation of the logarithmic derivative of the Selberg zeta function as a Gauss transform of the hyperbolic heat trace. By means of the spectral theoretical realization of the hyperbolic heat trace obtained by an application of the Selberg trace formula, we are able to derive a lower bound for the logarithmic derivative of the Selberg zeta function depending on a free parameter. By an optimal choice of this parameter we arrive at the lower bound for c_M depending only on the volume, the number of cusps and the total branching order of M.

In section 4, we obtain a first upper bound for c_M (see Theorem 4.7). For its proof we again make use of the realization of the logarithmic derivative of the Selberg zeta function as a Gauss transform of the hyperbolic heat trace, and then estimate the hyperbolic heat trace from above. This time, we use the prime geodesic theorem and techniques from analytic number theory to derive an upper bound for the hyperbolic heat trace in terms of small eigenvalues and short, primitive geodesics. The upper bound obtained for c_M has the option of using the small eigenvalues less than $\varepsilon < 1/4$; then, the number of short, primitive geodesics needed is explicitly determined as a function of ε and a positive constant C_M implied by the prime geodesic theorem for M. Under the additional assumption that M is a finite cover of some fixed quotient space $M_0 = \Gamma_0 \backslash \mathbb{H}$ we are able to give a second upper bound for c_M in terms of the volume of M and certain small eigenvalues of M only (see Theorem 4.11).

Finally, in section 5, we present some examples related to the standard congruence subgroups $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$ of $\mathrm{PSL}_2(\mathbb{Z})$ of square-free level N. In all three cases the main result of section 3 shows that the lower bound for c_M is of the order $O(\log N)$. On the other hand, an application of the results of section 4 together with results of M. Huxley on the number of conjugacy classes of matrices of bounded trace in congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ proves in all three cases that the upper bound for c_M is of the form $O_\varepsilon(N^\varepsilon)$ as mentioned in

1.1; in particular, we note that the growth of c_M is significantly slower than the growth of the volume of the corresponding Riemann surfaces.

2 Preliminaries

2.1. Riemann surfaces. Let $\mathbb{H}$ denote the upper half plane, $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ any Fuchsian group of the first kind and $M := \Gamma\backslash\mathbb{H}$ the associated quotient space, which can be given the structure of a Riemann surface. We denote by p_M, resp. n_M the number of cusps, resp. elliptic fixed points of M and by g_M the genus of the compact Riemann surface obtained from M by adding the cusps; we note that $p_M = 0$, if M is compact. We define the total branching order b_M of M by

$$b_M := \sum_{k=1}^{n_M}(m_k - 1)\,,$$

where m_k is the order of the corresponding elliptic element; we note that $b_M = 0$, if there are no elliptic elements in Γ. We recall that M is equipped with the complete hyperbolic metric of constant curvature -1, which is induced from the canonical hyperbolic metric on $\mathbb{H}$; we denote the corresponding hyperbolic volume element by $d\mu$. From [22], p. 42, we recall the formula

$$\frac{\mathrm{vol}(M)}{2\pi} = 2g_M - 2 + p_M + \sum_{k=1}^{n_M}\left(1 - \frac{1}{m_k}\right).$$

Using the well-known inequality $\mathrm{vol}(M)/2\pi \geq 1/42$ (cf. [22], p. 44), we easily derive from the above volume formula

$$p_M \leq \frac{85}{2\pi}\,\mathrm{vol}(M),\;\; n_M \leq \frac{85}{\pi}\,\mathrm{vol}(M).$$

2.2. The set $H(\Gamma)$. Let $H(\Gamma)$ denote a complete set of representatives of inconjugate, primitive, hyperbolic elements in Γ. For $\gamma \in H(\Gamma)$, we have $|\mathrm{tr}(\gamma)| > 2$; furthermore, we note that $\gamma \in H(\Gamma)$ generates its own centralizer in Γ. We denote by ℓ_γ the hyperbolic length of the closed geodesic determined by $\gamma \in H(\Gamma)$ on M; it is well-known that the equality

$$|\mathrm{tr}(\gamma)| = 2\cosh(\ell_\gamma/2)$$

holds.

2.3. Selberg zeta functions. For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, the Selberg zeta function $Z_M(s)$ associated to M is defined via the Euler product expansion

$$Z_M(s) := \prod_{\gamma \in H(\Gamma)} Z_\gamma(s),$$

where the local factors $Z_\gamma(s)$ are given by

$$Z_\gamma(s) := \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)\ell_\gamma}\right).$$

The Selberg zeta function $Z_M(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian (cf. [7], p. 498); furthermore, $Z_M(s)$ satisfies a functional equation. For our purposes, it suffices to know that the Selberg zeta function $Z_M(s)$ has a simple zero at $s = 1$. In particular, the quantity

$$c_M := \lim_{s \to 1} \left(\frac{Z_M'}{Z_M}(s) - \frac{1}{s-1} \right)$$

is well-defined.

2.4. Heat kernels. The heat kernel $K_{\mathbb{H}}(t; \tilde{x}, \tilde{y})$ on $\mathbb{H}$ ($t \in \mathbb{R}_{>0}$; $\tilde{x}, \tilde{y} \in \mathbb{H}$; $\tilde{x} \neq \tilde{y}$) is given by the formula

$$K_{\mathbb{H}}(t; \tilde{x}, \tilde{y}) = K_{\mathbb{H}}(t; \rho) := \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{re^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(\rho)}} \, dr,$$

where $\rho := d_{\mathbb{H}}(\tilde{x}, \tilde{y})$ denotes the hyperbolic distance between $\tilde{x}$ and $\tilde{y}$ (cf. [4], p. 246); if $\tilde{x} = \tilde{y}$, i.e., $\rho = 0$, the previous formula can be shown to equal

$$K_{\mathbb{H}}(t; \tilde{x}, \tilde{x}) = K_{\mathbb{H}}(t; 0) = \frac{1}{2\pi} \int_{0}^{\infty} e^{-(r^2+1/4)t} \, r \tanh(\pi r) \, dr.$$

The heat kernel $K_M(t; x, y)$ associated to M ($t \in \mathbb{R}_{>0}$; $x, y \in M$), resp. the hyperbolic heat kernel $HK_M(t; x)$ associated to M ($t \in \mathbb{R}_{>0}$; $x \in M$), are now defined by averaging over the elements of Γ, resp. the hyperbolic elements of Γ, namely (cf. [6], p. 347),

$$\begin{aligned} K_M(t; x, y) &:= \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t; \tilde{x}, \gamma(\tilde{y})), \text{ resp.} \\ HK_M(t; x) &:= \sum_{\substack{\gamma \in \Gamma \\ |\mathrm{tr}(\gamma)| > 2}} K_{\mathbb{H}}(t; \tilde{x}, \gamma(\tilde{x})); \end{aligned}$$

here $\tilde{x}$, resp. $\tilde{y}$ are (any) lifts of x, resp. y to $\mathbb{H}$.

2.5. Hyperbolic heat traces. If M is non-compact, the hyperbolic heat kernel $K_M(t; x, y)$ is not of trace class whereas the hyperbolic heat kernel $HK_M(t; x)$

is integrable with respect to x. With this in mind, we define as in [15] the hyperbolic heat trace $HTrK_M(t)$ associated to M ($t \in \mathbb{R}_{>0}$) by the integral

$$HTrK_M(t) := \int\limits_M HK_M(t;x)\, d\mu(x).$$

In analogy to the factorization of the Selberg zeta function $Z_M(s)$ into local factors, there is a decomposition of the hyperbolic heat trace $HTrK_M(t)$ into local constituents. Namely, we have

$$HTrK_M(t) = \sum_{\gamma \in H(\Gamma)} HTrK_\gamma(t),$$

where $HTrK_\gamma(t)$ is explicitly given by the expression

$$HTrK_\gamma(t) = \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=1}^{\infty} \frac{\ell_\gamma}{e^{n\ell_\gamma/2} - e^{-n\ell_\gamma/2}} e^{-(n\ell_\gamma)^2/4t}$$

(cf. [6], p. 349; [7], p. 207).
A spectral theoretical realization of the hyperbolic heat trace $HTrK_M(t)$ is obtained by an application of the Selberg trace formula. For this purpose, we let $\{\lambda_M\}$ denote the eigenvalues of the hyperbolic Laplacian acting on the space of smooth functions on M. There is only a finite number of eigenvalues λ_M satisfying $0 \leq \lambda_M < 1/4$; as usual, we label these eigenvalues by $0 = \lambda_{M,0} < \lambda_{M,1} \leq \lambda_{M,2} \leq ... < 1/4$. Furthermore, there is a measure $d\mu_M$ defined on $\mathbb{R}$, which describes the distribution of the eigenvalues $\lambda_M \geq 1/4$, in particular of those eigenvalues coming from the continuous spectrum (cf. [7], p. 315). With these notations, we recall the formula

$$\begin{aligned} &HTrK_M(t) + \mathrm{vol}(M) \cdot K_{\mathbb{H}}(t;0) = \\ &\sum_{0 \leq \lambda_{M,j} < 1/4} e^{-\lambda_{M,j} t} + \int\limits_0^\infty e^{-(r^2+1/4)t}\, d\mu_M(r). \end{aligned} \tag{1}$$

If M is compact, the measure $d\mu_M(r)$ is concentrated with mass one in the points of the form $r = \sqrt{\lambda_M - 1/4}$ with $\lambda_M \geq 1/4$.

2.6. McKean formulae. As a next ingredient, we recall the McKean formulae for the Selberg zeta function and its local factors (cf. [19] and [15], p. 656). For this purpose, we let $s \in \mathbb{C}$ with $\mathrm{Re}(s(s-1)) > 0$. Then, the McKean formula for the local factor $Z_\gamma(s)$ states the equality

$$\frac{Z'_\gamma}{Z_\gamma}(s) = (2s-1) \int\limits_0^\infty HTrK_\gamma(t) e^{-s(s-1)t}\, dt.$$

Accordingly, the McKean formula for $Z_M(s)$ becomes

$$\frac{Z_M'}{Z_M}(s) = \sum_{\gamma\in H(\Gamma)} \frac{Z_\gamma'}{Z_\gamma}(s) = (2s-1)\int\limits_0^\infty HTrK_M(t)e^{-s(s-1)t}\,dt.$$

2.7. Small eigenvalues. A small eigenvalue of the hyperbolic Laplacian is any eigenvalue, which is less than 1/4. For any $a \geq 0$ and $a < b \leq 1/4$, we let

$$N_{M;a,b} := \#\{\lambda_{M,j} \,|\, a \leq \lambda_{M,j} < b\}.$$

From [23], p. 1620 (cf. also [2]), we recall the upper bound

$$N_{M;0,1/4} \leq 4g_M + 2p_M + 2n_M - 2.$$

From this and the estimate

$$\begin{aligned} \frac{\mathrm{vol}(M)}{\pi} &= 4g_M - 4 + 2p_M + 2\sum_{k=1}^{n_M}\left(1 - \frac{1}{m_k}\right) \\ &\geq 4g_M - 4 + 2p_M + n_M, \end{aligned}$$

we easily deduce the alternative upper bound

$$N_{M;0,1/4} \leq \frac{128}{\pi}\,\mathrm{vol}(M),$$

using also $\mathrm{vol}(M) \geq 1/42$ and $n_M \leq 85/\pi \cdot \mathrm{vol}(M)$.

2.8. The prime geodesic counting function. Consider the function

$$\pi_M(u) := \sum_{\substack{\gamma\in H(\Gamma)\\ e^{\ell_\gamma} < u}} 1,$$

which is defined for $u \in \mathbb{R}_{>1}$; it is just the number of inconjugate, primitive, hyperbolic elements of Γ such that the corresponding geodesics have length less than $\log u$. Letting now $\ell_0 > 0$ denote a real number, which is smaller than the length of the shortest closed geodesic on M, and introducing the function

$$f(u,t) := \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=1}^{\infty} \frac{\log u}{u^{n/2} - u^{-n/2}} e^{-(n\log u)^2/4t},$$

we immediately derive from the decomposition of the hyperbolic heat trace into its local constituents the formula

$$HTrK_M(t) = \int\limits_{e^{\ell_0}}^{\infty} f(u,t)\,d\pi_M(u).$$

2.9. The prime geodesic theorem. For any $0 < \varepsilon \leq 1/4$, we define

$$s_\varepsilon := \frac{1}{2} + \sqrt{\frac{1}{4} - \varepsilon}\,;$$

furthermore, we put for any small eigenvalue $\lambda_{M,j}$,

$$s_{M,j} := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{M,j}}\,,$$

and note that $1/2 < s_{M,j} \leq 1$. Introducing the integral logarithm

$$\mathrm{li}(u^{s_{M,j}}) := \int\limits_{2}^{u^{s_{M,j}}} \frac{d\xi}{\log \xi}\,,$$

the prime geodesic theorem states

$$\pi_M(u) = \sum_{0 \leq \lambda_{M,j} < 1/4} \mathrm{li}(u^{s_{M,j}}) + O\left(u^{3/4}(\log u)^{-1/2}\right)$$

for $u > 2$, where the implied constant depends solely on M (cf. [8], p. 474).

3 A lower bound for c_M

3.1. Lemma. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ and M be as in section 2. Then, for any $\delta \geq 2$, we have the lower bound

$$HTrK_M(t) \geq \begin{cases} 0 & 0 < t \leq \delta, \\ 1 - \mathrm{vol}(M) K_{\mathbb{H}}(t;0) - \left(\dfrac{p_M}{2} + \dfrac{5b_M}{4} + \dfrac{N_{M;0,1/4}\delta_{p_M}}{2}\right) e^{-t/4} & t > \delta, \end{cases}$$

where $\delta_{p_M} = 1$, if $p_M > 0$, and $\delta_{p_M} = 0$, if $p_M = 0$.

Proof. By its definition, the hyperbolic heat trace $HTrK_M(t)$ is non-negative; in particular, $HTrK_M(t)$ is non-negative for all t, $0 < t \leq \delta$ for any $\delta \geq 2$. This proves the first part of the claim.
For the second part of the claim, we use the spectral theoretical realization of the hyperbolic heat trace obtained by an application of the Selberg trace formula.

Quoting from [9], Proposition 5.1, which is taken from [7], p. 315, we have

$$\begin{aligned}
HTrK_M(t) &= \sum_{\lambda_{M,j}\in D(M)} e^{-\lambda_{M,j}t} - \mathrm{vol}(M)K_{\mathbb{H}}(t;0) \\
&- \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-(r^2+1/4)t}\,\frac{\phi'}{\phi}(1/2+ir)\,dr \\
&+ \frac{p_M}{2\pi}\int_{-\infty}^{\infty} e^{-(r^2+1/4)t}\,\frac{\Gamma'}{\Gamma}(1+ir)\,dr \\
&- \frac{1}{4}\left(p_M - \mathrm{tr}\,\Phi(1/2)\right)e^{-t/4} + \frac{p_M\log 2}{\sqrt{4\pi t}}e^{-t/4} \\
&- \sum_{k=1}^{n_M}\sum_{n=1}^{m_k-1}\frac{1}{2m_k\sin(n\pi/m_k)}\int_{-\infty}^{\infty}\frac{e^{-2\pi nr/m_k}}{1+e^{-\pi r}}e^{-(r^2+1/4)t}\,dr,
\end{aligned}$$

where $D(M)$ denotes the set of L^2-eigenvalues of the hyperbolic Laplacian and ϕ is the determinant of the scattering matrix Φ. The claimed upper bound will follow from an analysis of the individual terms in the previous expression. Trivially, we have for the first summand

$$\sum_{\lambda_{M,j}\in D(M)} e^{-\lambda_{M,j}t} \geq 1,$$

since $\lambda_{M,0}=0$. Quoting [9], Lemma 5.3, which is taken from [7], line (12.4) and Proposition 12.7, we have for the third summand

$$-\frac{\phi'}{\phi}(1/2+ir) \geq \sum_{0\leq\lambda_{M,j}<1/4}\frac{(1-2s_{M,j})}{(s_{M,j}-1/2)^2+r^2} + 2\log q_M;$$

recalling that $1/2 < s_{M,j} \leq 1$ and $q_M > 1$ by [21], p. 16, we need to study

$$\begin{aligned}
&-\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-(r^2+1/4)t}\,\frac{\phi'}{\phi}(1/2+ir)\,dr \geq \\
&-\frac{1}{2\pi}\sum_{0\leq\lambda_{M,j}<1/4}\int_{-\infty}^{\infty} e^{-(r^2+1/4)t}\frac{(s_{M,j}-1/2)}{(s_{M,j}-1/2)^2+r^2}\,dr.
\end{aligned}$$

For $a>0$ consider the function

$$F_a(t) := \int_{-\infty}^{\infty}\frac{a}{a^2+r^2}e^{-r^2t}\,dr = \int_{-\infty}^{\infty}\frac{1}{1+u^2}e^{-a^2u^2t}\,du.$$

We immediately have for $t > 0$

$$0 \leq F_a(t) \leq F_a(0) = \pi,$$

and, by continuity, $F_0(t) = \pi$; therefore,

$$-\frac{1}{4\pi} \int\limits_{-\infty}^{\infty} e^{-(r^2+1/4)t} \, \frac{\phi'}{\phi}(1/2 + ir)\, dr \geq$$
$$-\frac{1}{2\pi} \sum_{0 \leq \lambda_{M,j} < 1/4} F_{s_{M,j}-1/2}(t) e^{-t/4} \geq -\frac{N_{M;0,1/4}}{2} e^{-t/4}.$$

Note that this term appears only, if $p_M > 0$, i.e., if $\delta_{p_M} = 1$. Using the Gauss formula for the Gamma function (cf. [13], p. 114), we have

$$\left| \frac{\Gamma'}{\Gamma}(1 + ir) \right| \leq \left| \frac{\Gamma'}{\Gamma}(1) \right| = \gamma,$$

where $\gamma = -\Gamma'(1)$ is the Euler constant; therefore, we find for the fourth summand

$$\frac{p_M}{2\pi} \int\limits_{-\infty}^{\infty} e^{-(r^2+1/4)t} \, \frac{\Gamma'}{\Gamma}(1 + ir)\, dr =$$
$$\frac{p_M}{2\pi} \int\limits_{-\infty}^{\infty} e^{-(r^2+1/4)t} \, \mathrm{Re} \left(\frac{\Gamma'}{\Gamma}(1 + ir) \right) \, dr \geq$$
$$-\frac{\gamma \, p_M}{2\pi} \int\limits_{-\infty}^{\infty} e^{-(r^2+1/4)t} \, dr = -\frac{\gamma \, p_M}{\sqrt{4\pi t}} \, e^{-t/4}.$$

The $(p_M \times p_M)$-matrix $\Phi(1/2)$ is symmetric and orthogonal (cf. [7], p. 305), so $-p_M \leq \mathrm{tr}\, \Phi(1/2) \leq p_M$; therefore, we obtain for the fifth summand

$$-\frac{1}{4} \left(p_M - \mathrm{tr}\, \Phi(1/2) \right) e^{-t/4} \geq -\frac{p_M}{2} e^{-t/4}.$$

Combining these computations together with the inequality $\log 2 > \gamma$, we get that the three terms in the spectral expansion for the hyperbolic heat trace involving p_M admit the lower bound

$$p_M \left(\frac{\log 2 - \gamma}{\sqrt{4\pi t}} - \frac{1}{2} \right) e^{-t/4} \geq -\frac{p_M}{2} e^{-t/4}.$$

It remains to bound the last summand, i.e., the contribution coming from the elliptic elements of Γ. To do so, let us write

$$\int\limits_{-\infty}^{\infty} \frac{e^{-2\pi n r/m_k} e^{-r^2 t}}{1 + e^{-\pi r}} \, dr = \int\limits_{-\infty}^{0} \frac{e^{-2\pi n r/m_k} e^{-r^2 t}}{1 + e^{-\pi r}} \, dr + \int\limits_{0}^{\infty} \frac{e^{-2\pi n r/m_k} e^{-r^2 t}}{1 + e^{-\pi r}} \, dr.$$

For the first integral, we have, since $0 < n < m_k$,

$$\int_{-\infty}^{0} \frac{e^{-2\pi nr/m_k} e^{-r^2 t}}{1+e^{-\pi r}}\, dr = \int_{0}^{\infty} \frac{e^{2\pi nr/m_k} e^{-r^2 t}}{1+e^{\pi r}}\, dr \leq \int_{0}^{\infty} \frac{e^{2\pi r} e^{-r^2 t}}{1+e^{\pi r}}\, dr \leq$$

$$\int_{0}^{\infty} e^{\pi r} e^{-r^2 t}\, dr \leq e^{\pi^2/4t} \int_{-\infty}^{\infty} e^{-(r\sqrt{t}-\pi/\sqrt{4t})^2}\, dr = \frac{\sqrt{\pi}}{\sqrt{t}}\, e^{\pi^2/4t}.$$

As for the second integral, we have

$$\int_{0}^{\infty} \frac{e^{-2\pi nr/m_k} e^{-r^2 t}}{1+e^{-\pi r}}\, dr \leq \int_{0}^{\infty} e^{-r^2 t}\, dr = \frac{\sqrt{\pi}}{\sqrt{4t}}.$$

Therefore, because $t > \delta \geq 2$, we have

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi nr/m_k} e^{-r^2 t}}{1+e^{-\pi r}}\, dr \leq \frac{\sqrt{\pi}}{\sqrt{t}} e^{\pi^2/4t} + \frac{\sqrt{\pi}}{\sqrt{4t}} \leq \frac{\sqrt{\pi}}{\sqrt{2}} e^{\pi^2/8} + \frac{\sqrt{\pi}}{\sqrt{8}} \leq 5.$$

Hence, the contribution from the elliptic elements is bounded by

$$\sum_{k=1}^{n_M} \sum_{n=1}^{m_k-1} \frac{1}{2m_k \sin(n\pi/m_k)} \int_{-\infty}^{\infty} \frac{e^{-2\pi nr/m_k}}{1+e^{-\pi r}} e^{-(r^2+1/4)t}\, dr \leq$$

$$\frac{5e^{-t/4}}{2} \sum_{k=1}^{n_M} \sum_{n=1}^{m_k-1} \frac{1}{m_k \sin(n\pi/m_k)}.$$

Let us use the inequalities $\sin(n\pi/m_k) \geq \sin(\pi/m_k)$ $(n = 1, ..., m_k - 1)$ and $m_k \sin(\pi/m_k) \geq 2$ for $m_k \geq 2$, to get the upper bound

$$\frac{5e^{-t/4}}{2} \sum_{k=1}^{n_M} \sum_{n=1}^{m_k-1} \frac{1}{m_k \sin(n\pi/m_k)} \leq \frac{5e^{-t/4}}{2} \sum_{k=1}^{n_M} \frac{m_k - 1}{m_k \sin(\pi/m_k)} \leq$$

$$\frac{5e^{-t/4}}{4} \sum_{k=1}^{n_M} (m_k - 1) = \frac{5b_M}{4} e^{-t/4}.$$

With all this, the proof of the lemma is complete. □

3.2. Lemma. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$, M and c_M be as in section 2. Then, for any $\delta \geq 2$, we have the lower bound

$$c_M \geq 1 - \delta - \mathrm{vol}(M) \int_{\delta}^{\infty} K_{\mathbb{H}}(t;0)\, dt - (2p_M + 5b_M + 2N_{M;0,1/4}\delta_{p_M}) e^{-\delta/4}.$$

Proof. By means of the McKean formula and Lemma 3.1, we derive for $s \in \mathbb{R}$, $s > 1$, the estimate

$$\frac{Z'_M}{Z_M}(s) = (2s-1)\int\limits_0^\infty HTrK_M(t)e^{-s(s-1)t}\,dt \geq (2s-1)\times$$
$$\int\limits_\delta^\infty \left(1 - \mathrm{vol}(M)K_{\mathbb{H}}(t;0) - \left(\frac{p_M}{2} + \frac{5b_M}{4} + \frac{N_{M;0,1/4}\delta_{p_M}}{2}\right)e^{-t/4}\right)e^{-s(s-1)t}\,dt =$$
$$\left(\frac{1}{s} + \frac{1}{s-1}\right)e^{-s(s-1)\delta} - (2s-1)\mathrm{vol}(M)\int\limits_\delta^\infty K_{\mathbb{H}}(t;0)e^{-s(s-1)t}\,dt -$$
$$\left(\frac{p_M}{2} + \frac{5b_M}{4} + \frac{N_{M;0,1/4}\delta_{p_M}}{2}\right)\frac{2s-1}{(s-1/2)^2}\,e^{-(s-1/2)^2\delta}.$$

Since we have for $s \in \mathbb{R}$, $s > 1$, $s \to 1$,

$$\frac{e^{-s(s-1)\delta}}{s-1} - \frac{1}{s-1} = -s\delta + O\left(s^2(s-1)\delta^2\right),$$

we arrive at the inequality

$$\begin{aligned} c_M &= \lim_{s\to 1}\left(\frac{Z'_M}{Z_M}(s) - \frac{1}{s-1}\right) \\ &\geq 1 - \delta - \mathrm{vol}(M)\int\limits_\delta^\infty K_{\mathbb{H}}(t;0)\,dt - (2p_M + 5b_M + 2N_{M;0,1/4}\delta_{p_M})e^{-\delta/4}. \end{aligned}$$

This completes the proof of the lemma. □

3.3. Theorem. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$, M and c_M be as in section 2. Then, we have the lower bound

$$c_M \geq -4\log\left(\frac{\mathrm{vol}(M)}{2\pi} + 2p_M + 5b_M + 2N_{M;0,1/4}\delta_{p_M}\right).$$

Proof. Since $\tanh(\pi r) \leq 1$ for all $r \in \mathbb{R}$, the heat kernel $K_{\mathbb{H}}(t;0)$ can be bounded as

$$K_{\mathbb{H}}(t;0) = \frac{1}{2\pi}\int\limits_0^\infty e^{-(r^2+1/4)t}\,r\tanh(\pi r)\,dr \leq \frac{e^{-t/4}}{2\pi}\int\limits_0^\infty e^{-r^2t}\,r\,dr = \frac{e^{-t/4}}{4\pi t}.$$

Since $\delta \geq 2$, we therefore obtain the upper bound

$$\int\limits_\delta^\infty K_{\mathbb{H}}(t;0)\,dt \leq \int\limits_\delta^\infty \frac{e^{-t/4}}{4\pi t}\,dt \leq \frac{e^{-\delta/4}}{\pi\delta} \leq \frac{e^{-\delta/4}}{2\pi},$$

which yields the inequality

$$c_M \geq 1 - \delta - \left(\frac{\mathrm{vol}(M)}{2\pi} + 2p_M + 5b_M + 2N_{M;0,1/4}\delta_{p_M} \right) e^{-\delta/4}.$$

This lower bound is optimized by taking

$$\delta = \delta_0 := 4 \log \left(\frac{\mathrm{vol}(M)}{2\pi} + 2p_M + 5b_M + 2N_{M;0,1/4}\delta_{p_M} \right),$$

which gives $c_M \geq -\delta_0$. Noting that any quotient $M = \Gamma\backslash\mathbb{H}$ has the property that $\delta_0 \geq 2$, then completes the proof of the theorem. □

3.4. Remarks. (i) Throughout the proofs of Lemma 3.1 and Theorem 3.3 we have simplified the lower bounds by eliminating negative powers of t. Upon further review, one sees that the lower bound in Lemma 3.1 can be improved to replace the summand containing $e^{-t/4}$ by a summand of the form $a_M t^{-1/4} e^{-t/4}$, where the coefficient a_M depends on p_M, b_M and $N_{M;0,1/4}\delta_{p_M}$. The resulting improvement in Theorem 3.3 will be that the lower bound will contain a second, additive term comparable to a positive multiple of $\log \delta_0$.

(ii) Some improvement in the lower bound can be obtained by keeping more of the terms from the sum over the discrete spectrum. Specifically, if we use the lower bound

$$\sum_{\lambda_{M,j} \in D(M)} e^{-\lambda_{M,j} t} \geq 1 + \sum_{0 < \lambda_{M,j} < 1/4} e^{-\lambda_{M,j} t}$$

for $t > \delta$ in Lemma 3.1, then the lower bound in Lemma 3.2 is improved by the additive term

$$\sum_{0 < \lambda_{M,j} < 1/4} \frac{1}{\lambda_{M,j}} e^{-\lambda_{M,j}\delta}.$$

With the same choice of δ_0 as in the proof of Theorem 3.3, the lower bound is then improved by the additive term

$$\sum_{0 < \lambda_{M,j} < 1/4} \frac{1}{\lambda_{M,j}} \left(\frac{\mathrm{vol}(M)}{2\pi} + 2p_M + 5b_M + 2N_{M;0,1/4}\delta_{p_M} \right)^{-4\lambda_{M,j}},$$

which is indeed a substantial improvement, if $\lambda_{M,1}$ is very small.

3.5. Remark. Recalling the upper bounds for p_M from 2.1 and $N_{M;0,1/4}$ from 2.7, we derive from Theorem 3.3 the lower bound

$$c_M \geq -4 \log \left(\frac{683}{2\pi} \mathrm{vol}(M) + 5b_M \right).$$

Furthermore, if $\{M\}$ is a sequence of quotient spaces associated to Fuchsian groups of the first kind such that the volume of M tends to infinity, but such that the orders m_k ($k = 1, ..., n_M$) of the elliptic fixed points of M are uniformly bounded by m_0, we obtain the lower bound

$$c_M \geq -4 \log\left(\frac{683 + 850m_0}{2\pi}\,\mathrm{vol}(M)\right),$$

using the estimate $b_M \leq m_0 \cdot n_M$.

4 An upper bound for c_M

4.1. Definition. For $0 < \varepsilon < 1/4$ and $\delta > 0$, the (ε, δ)-truncated hyperbolic heat trace $HTrK_{M,\varepsilon,\delta}(t)$ is defined by

$$HTrK_{M,\varepsilon,\delta}(t) := HTrK_M(t) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} e^{-\lambda_{M,j} t} - \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} HTrK_\gamma(t).$$

4.2. Lemma. By means of the (ε, δ)-truncated hyperbolic heat trace, the invariant c_M can be expressed as

$$c_M = 1 + \sum_{0 < \lambda_{M,j} < \varepsilon} \frac{1}{\lambda_{M,j}} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} \frac{Z'_\gamma}{Z_\gamma}(1) + \int\limits_0^\infty HTrK_{M,\varepsilon,\delta}(t)\, dt.$$

Proof. We first multiply the equality

$$HTrK_M(t) = \sum_{0 \leq \lambda_{M,j} < \varepsilon} e^{-\lambda_{M,j} t} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} HTrK_\gamma(t) + HTrK_{M,\varepsilon,\delta}(t)$$

by $e^{-s(s-1)t}$, $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and then integrate the resulting equality along t from 0 to ∞. After multiplying by $(2s - 1)$, we obtain

$$\begin{aligned} \frac{Z'_M}{Z_M}(s) &= (2s-1) \sum_{0 \leq \lambda_{M,j} < \varepsilon} \frac{1}{s(s-1) + \lambda_{M,j}} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} \frac{Z'_\gamma}{Z_\gamma}(s) \\ &+ (2s-1) \int\limits_0^\infty HTrK_{M,\varepsilon,\delta}(t) e^{-s(s-1)t}\, dt. \end{aligned}$$

Observing the equality

$$\sum_{0 \leq \lambda_{M,j} < \varepsilon} \frac{2s-1}{s(s-1) + \lambda_{M,j}} = \frac{1}{s} + \frac{1}{s-1} + \sum_{0 < \lambda_{M,j} < \varepsilon} \frac{2s-1}{s(s-1) + \lambda_{M,j}}\ ,$$

the claim of the lemma follows from the previous equation after letting s go to 1, taking also into account that the sum over $\gamma \in H(\Gamma)$ with $\ell_\gamma < \delta$ is finite and that the function $HTrK_{M,\varepsilon,\delta}(t)$ is integrable, since it decays exponentially both near $t = 0$ and near $t = \infty$ (cf. [15], pp. 647-648). $\square$

4.3. Lemma. Let $0 < \varepsilon < 1/4$. Then, the following integral inequality for the function $f(u,t)$ of 2.8 holds for $t \in \mathbb{R}_{>0}$

$$\int\limits_{e^\delta}^{\infty} f(u,t)\, d\mathrm{li}(u^{s_{M,j}}) \leq e^{-\lambda_{M,j}t} + C(\delta) e^{(s_{M,j}-1)\delta} \sqrt{t} e^{-t/4} e^{-\delta^2/4t},$$

where

$$C(\delta) = \frac{\pi^{3/2}}{3\delta} \left(\frac{e^{-\delta/2}}{1-e^{-\delta}} + 1 \right).$$

Proof. Recalling the definition of the function $f(u,t)$ from 2.8 and the formula

$$d\mathrm{li}(u^{s_{M,j}}) = \frac{u^{s_{M,j}-1}}{\log u}\, du\ ,$$

we obtain

$$\int\limits_{e^\delta}^{\infty} f(u,t) d\mathrm{li}(u^{s_{M,j}}) = \frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{e^\delta}^{\infty} \sum_{n=1}^{\infty} \frac{u^{s_{M,j}}}{u^{n/2} - u^{-n/2}} e^{-(n \log u)^2/4t}\, \frac{du}{u} =$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{\delta}^{\infty} \sum_{n=1}^{\infty} \frac{e^{s_{M,j}\xi}}{e^{n\xi/2} - e^{-n\xi/2}} e^{-(n\xi)^2/4t}\, d\xi =$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{\delta}^{\infty} \sum_{n=1}^{\infty} \frac{e^{s_{M,j}\xi - n\xi/2}}{1 - e^{-n\xi}} e^{-(n\xi)^2/4t}\, d\xi =$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{\delta}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} e^{(s_{M,j} - n/2 - nm)\xi} e^{-(n\xi)^2/4t}\, d\xi.$$

In order to estimate the latter integral from above, we decompose the double sum into three summands, namely the terms arising for $m = 0, n = 1$, for $m = 0, n \geq 2$ and for $m, n \geq 1$. For the summand belonging to $m = 0, n = 1$, an easy computation yields

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{\delta}^{\infty} e^{(s_{M,j}-1/2)\xi} e^{-\xi^2/4t}\, d\xi =$$

$$\frac{e^{-\lambda_{M,j}t}}{(4\pi t)^{1/2}} \int\limits_{\delta}^{\infty} e^{-(\xi/\sqrt{4t} - (s_{M,j}-1/2)\sqrt{t})^2}\, d\xi =$$

$$\frac{e^{-\lambda_{M,j}t}}{\sqrt{\pi}} \left(\int\limits_{-\infty}^{\infty} e^{-\eta^2}\, d\eta - \int\limits_{-\infty}^{\delta/\sqrt{4t} - (s_{M,j}-1/2)\sqrt{t}} e^{-\eta^2}\, d\eta \right) \leq e^{-\lambda_{M,j}t}.$$

For the summand belonging to $m = 0, n \geq 2$, we next obtain the estimate observing that $s_{M,j} - n/2 \leq 0$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int_\delta^\infty \sum_{n=2}^\infty e^{(s_{M,j}-n/2)\xi} e^{-(n\xi)^2/4t}\, d\xi \leq$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=2}^\infty e^{(s_{M,j}-n/2)\delta} \int_\delta^\infty e^{-(n\xi)^2/4t}\, d\xi \leq$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=2}^\infty e^{(s_{M,j}-n/2)\delta} \frac{4t}{n^2\delta} e^{-(n\delta)^2/4t} \leq$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=2}^\infty e^{(s_{M,j}-1)\delta} \frac{4t}{n^2\delta} e^{-\delta^2/4t} \leq$$

$$\frac{2}{\sqrt{\pi}\delta} e^{(s_{M,j}-1)\delta} \sqrt{t} e^{-t/4} e^{-\delta^2/4t} \sum_{n=2}^\infty \frac{1}{n^2} \leq$$

$$\frac{\pi^{3/2}}{3\delta} e^{(s_{M,j}-1)\delta} \sqrt{t} e^{-t/4} e^{-\delta^2/4t}.$$

Finally, for the summand belonging to $m, n \geq 1$, we obtain in a similar way as above the estimate

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int_\delta^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty e^{(s_{M,j}-n/2-nm)\xi} e^{-(n\xi)^2/4t}\, d\xi =$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int_\delta^\infty \sum_{n=1}^\infty \frac{e^{(s_{M,j}-3n/2)\xi}}{1-e^{-n\xi}} e^{-(n\xi)^2/4t}\, d\xi \leq$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=1}^\infty \frac{e^{(s_{M,j}-3n/2)\delta}}{1-e^{-n\delta}} \int_\delta^\infty e^{-(n\xi)^2/4t}\, d\xi \leq$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{n=1}^\infty \frac{e^{(s_{M,j}-3n/2)\delta}}{1-e^{-n\delta}} \cdot \frac{4t}{n^2\delta} e^{-(n\delta)^2/4t} \leq$$

$$\frac{2}{\sqrt{\pi}\delta} \cdot \frac{e^{(s_{M,j}-3/2)\delta}}{1-e^{-\delta}} \sqrt{t} e^{-t/4} e^{-\delta^2/4t} \sum_{n=1}^\infty \frac{1}{n^2} =$$

$$\frac{\pi^{3/2}}{3\delta} \cdot \frac{e^{-\delta/2}}{1-e^{-\delta}} e^{(s_{M,j}-1)\delta} \sqrt{t} e^{-t/4} e^{-\delta^2/4t}.$$

The proof of the lemma follows now immediately from the last three estimates. □

4.4. Lemma. For the function $f(u,t)$ of 2.8 we have the following integral formula for $u \in \mathbb{R}_{>1}$

$$\int_0^\infty f(u,t)\,dt = \sum_{n=1}^\infty \frac{\log u}{u^n - 1}.$$

Secondly, we have the integral formula

$$\int_0^\infty \sqrt{t} e^{-t/4} e^{-\delta^2/4t}\,dt = \sqrt{\pi}(2\delta + 4)e^{-\delta/2}.$$

Proof. For $v, w \in \mathbb{R}_{>0}$ and $\sigma \in \mathbb{R}$ we recall the K-Bessel function (cf. [14], p. 101)

$$K_\sigma(v,w) := \int_0^\infty t^{\sigma-1} e^{-v^2 t} e^{-w^2/t}\,dt.$$

In particular, we recall the formulae for $\sigma = 1/2$, resp. $\sigma = 3/2$,

$$K_{1/2}(v,w) = \frac{\sqrt{\pi}}{v} e^{-2vw} \text{ , resp.}$$

$$K_{3/2}(v,w) = -\frac{1}{2v}\frac{d}{dv} K_{1/2}(v,w) = \frac{\sqrt{\pi}}{2} \cdot \frac{2vw+1}{v^3} e^{-2vw}.$$

From the definition of the function $f(u,t)$ we now obtain for the first integral by means of the formula for $K_{1/2}(v,w)$

$$\int_0^\infty f(u,t)\,dt = \int_0^\infty \frac{1}{\sqrt{4\pi}} \sum_{n=1}^\infty \frac{\log u}{u^{n/2} - u^{-n/2}}\, t^{-1/2} e^{-t/4} e^{-(n \log u)^2/4t}\,dt =$$

$$\frac{1}{\sqrt{4\pi}} \sum_{n=1}^\infty \frac{\log u}{u^{n/2} - u^{-n/2}}\, K_{1/2}(1/2, n/2 \log u) =$$

$$\sum_{n=1}^\infty \frac{\log u}{u^{n/2} - u^{-n/2}}\, e^{-n/2 \log u} = \sum_{n=1}^\infty \frac{\log u}{u^n - 1}\ ,$$

as claimed. To establish the second integral formula, we simply have to note that

$$\int_0^\infty \sqrt{t} e^{-t/4} e^{-\delta^2/4t}\,dt = K_{3/2}(1/2, \delta/2) = \sqrt{\pi}(2\delta + 4)e^{-\delta/2}$$

by means of the formula for $K_{3/2}(v,w)$. $\square$

4.5. Proposition. Let $0 < \varepsilon < 1/4$, $\delta > 4$. Then, the following inequality holds

$$\int_0^\infty HTrK_{M,\varepsilon,\delta}(t)\,dt \leq$$

$$\int_{e^\delta}^\infty \sum_{n=1}^\infty \frac{\log u}{u^n - 1}\, d\left[\pi_M(u) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}})\right] + 12 N_{M;0,\varepsilon} e^{-\delta/2}.$$

Proof. Recalling the definition of the (ε, δ)-truncated hyperbolic heat trace

$$HTrK_{M,\varepsilon,\delta}(t) = HTrK_M(t) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} e^{-\lambda_{M,j} t} - \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} HTrK_\gamma(t),$$

we obtain by means of the Stieltjes integral in 2.8 and Lemma 4.3

$$\begin{aligned} HTrK_{M,\varepsilon,\delta}(t) \quad &\leq \quad \int_{e^\delta}^\infty f(u,t)\, d\pi_M(u) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} \int_{e^\delta}^\infty f(u,t)\, d\mathrm{li}(u^{s_{M,j}}) \\ &+ \quad \sum_{0 \leq \lambda_{M,j} < \varepsilon} C(\delta) e^{(s_{M,j} - 1)\delta} \sqrt{t} e^{-t/4} e^{-\delta^2/4t}. \end{aligned}$$

We now integrate this inequality from $t = 0$ to $t = \infty$. By interchanging the order of integrations, which is permitted by the bound given by the prime geodesic theorem, and taking into account Lemma 4.4, we find

$$\begin{aligned} \int_0^\infty HTrK_{M,\varepsilon,\delta}(t)\,dt \quad &\leq \quad \int_{e^\delta}^\infty \sum_{n=1}^\infty \frac{\log u}{u^n - 1}\, d\left[\pi_M(u) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}})\right] \\ &+ \quad \sum_{0 \leq \lambda_{M,j} < \varepsilon} \sqrt{\pi}(2\delta + 4) C(\delta) e^{(s_{M,j} - 3/2)\delta}. \end{aligned}$$

Taking $\delta > 4 > \log 54$, we find the estimate

$$\sqrt{\pi}(2\delta + 4) C(\delta) = \pi^2 \left(\frac{2}{3} + \frac{4}{3\delta}\right)\left(\frac{e^{-\delta/2}}{1 - e^{-\delta}} + 1\right) \leq$$

$$\pi^2 \left(\frac{2}{3} + \frac{4}{3 \cdot 4}\right)\left(\frac{1/\sqrt{54}}{1 - e^{-\log 54}} + 1\right) \leq \frac{61}{53}\pi^2 < 12.$$

Observing finally that $s_{M,j} \leq 1$, i.e., $s_{M,j} - 3/2 \leq -1/2$, and taking into account the last two estimates, we arrive at the inequality

$$\int_0^\infty HTrK_{M,\varepsilon,\delta}(t)\,dt \leq$$
$$\int_{e^\delta}^\infty \sum_{n=1}^\infty \frac{\log u}{u^n-1}\, d\left[\pi_M(u) - \sum_{0\leq\lambda_{M,j}<\varepsilon} \mathrm{li}(u^{s_{M,j}})\right] + \sum_{0\leq\lambda_{M,j}<\varepsilon} 12e^{-\delta/2},$$

from which the proof of the proposition follows. □

4.6. Lemma. Let F be a real valued, smooth, decreasing function defined for $u > 0$ and let g_1, g_2 be real valued, non decreasing functions defined for $u \geq a > 0$ and satisfying $g_1(u) \leq g_2(u)$ for $u \geq a$. Then, the following inequality of Stieltjes integrals

$$\int_a^\infty F(u)\,dg_1(u) + F(a)\,g_1(a) \leq \int_a^\infty F(u)\,dg_2(u) + F(a)\,g_2(a)$$

holds, provided both integrals exist.

Proof. If $g_1(a) = g_2(a)$ holds, the proof of the stated inequality is an immediate consequence of Lemma 2.2 in [18]. If $g_1(a) < g_2(a)$, we use the following adaption to Lundelius's argument. For any b, $0 < b < a$, we define

$$\tilde{g}_1(u) := \begin{cases} g_1(a) & b \leq u < a, \\ g_1(u) & u \geq a, \end{cases}$$

and

$$\tilde{g}_2(u) := \begin{cases} \dfrac{g_2(a) - g_1(a)}{a-b} \cdot (u-b) + g_1(a) & b \leq u < a, \\ g_2(u) & u \geq a. \end{cases}$$

By construction, the real valued functions $\tilde{g}_1, \tilde{g}_2$ are again non decreasing, satisfy $\tilde{g}_1(u) \leq \tilde{g}_2(u)$ for all $u \geq b$ and $\tilde{g}_1(b) = \tilde{g}_2(b)$. Another application of Lemma 2.2 of [18] then leads to the inequality

$$\int_b^\infty F(u)\,d\tilde{g}_1(u) \leq \int_b^\infty F(u)\,d\tilde{g}_2(u).$$

Since we have

$$d\tilde{g}_1(u) = 0 \quad \text{and} \quad d\tilde{g}_2(u) = \frac{g_2(a) - g_1(a)}{a-b}\,du$$

for $b \leq u < a$, we obtain the following inequality for all b, $0 < b < a$,

$$\int\limits_a^\infty F(u)\,dg_1(u) \leq \frac{g_2(a) - g_1(a)}{a-b} \int\limits_b^a F(u)\,du + \int\limits_a^\infty F(u)\,dg_2(u).$$

By letting b approach a and observing

$$\lim_{b \to a} \left(\frac{1}{a-b} \int\limits_b^a F(u)\,du \right) = F(a),$$

we now easily conclude the proof of the lemma by means of the previous inequality. $\square$

4.7. Theorem. Let $0 < \varepsilon \leq 7/64$. Then, with the above notations, there exists an explicit $\delta = \delta_{M,\varepsilon}$ depending on M and ε, such that the following upper bound holds

$$c_M \leq 3 + \sum_{0 < \lambda_{M,j} < \varepsilon} \frac{1}{\lambda_{M,j}} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1).$$

Proof. By applying Lemma 4.2 together with Proposition 4.5 we obtain the inequality

$$\begin{aligned} c_M \quad \leq \quad & 1 + \sum_{0 < \lambda_{M,j} < \varepsilon} \frac{1}{\lambda_{M,j}} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} \frac{Z'_\gamma}{Z_\gamma}(1) \\ + & \int\limits_{e^\delta}^\infty \sum_{n=1}^\infty \frac{\log u}{u^n - 1}\, d\left[\pi_M(u) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}}) \right] + 12 N_{M;0,\varepsilon} e^{-\delta/2}. \end{aligned}$$

As a next step, we are going to bound the remaining integral. For this we use the prime geodesic theorem, which was recalled in 2.9. It states that there is a constant $C_M > 0$ such that the inequalities

$$-C_M \cdot u^{3/4} (\log u)^{-1/2} \leq \pi_M(u) - \sum_{0 \leq \lambda_{M,j} < 1/4} \mathrm{li}(u^{s_{M,j}}) \leq C_M \cdot u^{3/4} (\log u)^{-1/2}$$

hold for $u > 2$. Since $\varepsilon \leq 7/64$, $s_\varepsilon \geq 7/8$, and we easily show that then the elementary inequality

$$\mathrm{li}(u^{s_\varepsilon}) \geq u^{3/4} (\log u)^{-1/2}$$

holds for $u > e^4$ and, hence, for $u \geq e^\delta$. Therefore, we derive from the prime geodesic theorem the inequalities

$$-C'_M \cdot \mathrm{li}(u^{s_\varepsilon}) \leq \pi_M(u) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}}) \leq C'_M \cdot \mathrm{li}(u^{s_\varepsilon}), \tag{2}$$

where $u > e^4$ and the implied constant C'_M is given by

$$C'_M = C_M + N_{M;\varepsilon,1/4}.$$

Now we apply Lemma 4.6 with $a = e^\delta$ and the functions

$$\begin{aligned}
F(u) &:= \sum_{n=1}^{\infty} \frac{\log u}{u^n - 1} \ , \\
g_1(u) &:= \pi_M(u) \ , \\
g_2(u) &:= \sum_{0 \le \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}}) + C'_M \cdot \mathrm{li}(u^{s_\varepsilon}) \ .
\end{aligned}$$

We note that the hypotheses of Lemma 4.6 are fulfilled, since $a > 0$, since F is a real valued, smooth, decreasing function defined for $u > e$ and since g_1, g_2 are real valued, non decreasing functions defined for $u \ge a$ and satisfying $g_1(u) \le g_2(u)$ for $u \ge a$ by the upper bound given in (2). The application of Lemma 4.6, after rearranging terms, then gives the inequality

$$\int_{e^\delta}^{\infty} F(u)\, d\left[\pi_M(u) - \sum_{0 \le \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}})\right] \le C'_M \int_{e^\delta}^{\infty} F(u)\, d\mathrm{li}(u^{s_\varepsilon}) +$$
$$F(e^\delta)\left(-\pi_M(e^\delta) + \sum_{0 \le \lambda_{M,j} < \varepsilon} \mathrm{li}(e^{s_{M,j}\delta}) + C'_M \cdot \mathrm{li}(e^{s_\varepsilon \delta})\right) .$$

Using now the lower bound given in (2), the above inequality can be rewritten as

$$\int_{e^\delta}^{\infty} F(u)\, d\left[\pi_M(u) - \sum_{0 \le \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}})\right] \le$$
$$C'_M \int_{e^\delta}^{\infty} F(u)\, d\mathrm{li}(u^{s_\varepsilon}) + 2C'_M F(e^\delta)\, \mathrm{li}(e^{s_\varepsilon \delta}).$$

Since $\delta > 4 > \log 2$, we have $u \ge e^\delta > 2$; therefore, we have the estimate

$$F(u) = \sum_{n=1}^{\infty} \frac{\log u}{u^n - 1} \le \sum_{n=1}^{\infty} \frac{2 \log u}{u^n} \le \frac{2 \log u}{u - 1} \le \frac{4 \log u}{u}.$$

This gives the upper bound

$$\begin{aligned}
&C'_M \int_{e^\delta}^{\infty} F(u)\, d\mathrm{li}(u^{s_\varepsilon}) + 2C'_M F(e^\delta)\, \mathrm{li}(e^{s_\varepsilon \delta}) \leq \\
&4C'_M \int_{e^\delta}^{\infty} \frac{\log u}{u}\, d\mathrm{li}(u^{s_\varepsilon}) + 8C'_M \frac{\log(e^\delta)}{e^\delta}\, \mathrm{li}(e^{s_\varepsilon \delta}) = \\
&\frac{4C'_M}{1-s_\varepsilon}\, e^{-(1-s_\varepsilon)\delta} + \frac{8C'_M \delta}{e^\delta}\, \mathrm{li}(e^{s_\varepsilon \delta}).
\end{aligned}$$

Observing now the elementary inequality

$$\mathrm{li}(u) \leq \frac{2u}{\log u}\,,$$

which holds for $u > e^2$ and implies

$$\frac{\delta}{e^\delta}\, \mathrm{li}(e^{s_\varepsilon \delta}) \leq \frac{2}{s_\varepsilon}\, e^{-(1-s_\varepsilon)\delta},$$

we finally obtain the inequality

$$\int_{e^\delta}^{\infty} F(u)\, d\left[\pi_M(u) - \sum_{0 \leq \lambda_{M,j} < \varepsilon} \mathrm{li}(u^{s_{M,j}})\right] \leq C_{M,\varepsilon} e^{-(1-s_\varepsilon)\delta}$$

with

$$C_{M,\varepsilon} := \frac{4(4-3s_\varepsilon)}{s_\varepsilon(1-s_\varepsilon)} \cdot C'_M = \frac{4(4-3s_\varepsilon)}{\varepsilon} \cdot C'_M = \frac{4(4-3s_\varepsilon)}{\varepsilon}(C_M + N_{M;\varepsilon,1/4})\,.$$

Returning to estimating the quantity c_M, we immediately derive from the preceeding bound

$$c_M \leq 1 + \sum_{0 < \lambda_{M,j} < \varepsilon} \frac{1}{\lambda_{M,j}} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} \frac{Z'_\gamma}{Z_\gamma}(1) + C_{M,\varepsilon} e^{-(1-s_\varepsilon)\delta} + 12 N_{M;0,\varepsilon} e^{-\delta/2}.$$

By finally choosing $\delta = \delta_{M,\varepsilon}$ with

$$\delta_{M,\varepsilon} := \max\left\{\frac{s_\varepsilon}{\varepsilon} \log(C_{M,\varepsilon}), 5 + 2\log(N_{M;0,\varepsilon})\right\},$$

we conclude the proof of the theorem by observing again $s_\varepsilon(1-s_\varepsilon) = \varepsilon$. Note that the previous requirement that $\delta > 4$ has been taken into account. □

4.8. Corollary. Taking $\varepsilon = 7/64$, we have the upper bound

$$c_M \leq 3 + \sum_{0<\lambda_{M,j}<\varepsilon} \frac{1}{\lambda_{M,j}} + \sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta_M}} \frac{Z'_\gamma}{Z_\gamma}(1),$$

where

$$\delta_M := \max\{32 + 8\log(C_M + N_{M;\varepsilon,1/4}), 5 + 2\log(N_{M;0,\varepsilon})\}.$$

In particular, if Γ is arithmetic, then there are no non-zero eigenvalues less than $7/64$ and, hence, we obtain an upper bound solely determined by the length spectrum less than $\delta_M = 32 + 8\log(C_M + N_{M;3/16,1/4})$.

Proof. The corollary is an immediate consequence of Theorem 4.7 applied with $\varepsilon = 7/64$, taking into account $s_\varepsilon = 7/8$. The rest follows by computing the first term in the formula given for $\delta_{M,\varepsilon}$. □

4.9. A revised lower bound. In this section we will briefly discuss how the tools developed in this section also apply to give a lower bound for c_M; for this we assume $0 < \varepsilon \leq 7/64$. We begin by recalling from Lemma 4.3 the formula

$$\int_{e^\delta}^{\infty} f(u,t) d\mathrm{li}(u^{s_{M,j}}) =$$
$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int_{\delta}^{\infty} \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{(s_{M,j}-n/2-nm)\xi} e^{-(n\xi)^2/4t}\, d\xi.$$

Omitting the terms corresponding to m, resp. n, which are different from $m = 0$, resp. $n = 1$, we obtain the estimate after rearranging terms

$$\begin{aligned} -e^{-\lambda_{M,j}t} \geq & - \int_{e^\delta}^{\infty} f(u,t) d\mathrm{li}(u^{s_{M,j}}) \\ & - \frac{e^{-t/4}}{(4\pi t)^{1/2}} \int_{-\infty}^{\delta} e^{(s_{M,j}-1/2)\xi} e^{-\xi^2/4t}\, d\xi. \end{aligned}$$

This immediately leads to the lower bound

$$\begin{aligned} HTrK_{M,\varepsilon,\delta}(t) \geq & \int_{e^\delta}^{\infty} f(u,t)\, d\left[\pi_M(u) - \sum_{0\leq\lambda_{M,j}<\varepsilon} \mathrm{li}(u^{s_{M,j}})\right] \\ & - \sum_{0\leq\lambda_{M,j}<\varepsilon} \frac{e^{-t/4}}{(4\pi t)^{1/2}} \int_{-\infty}^{\delta} e^{(s_{M,j}-1/2)\xi} e^{-\xi^2/4t}\, d\xi. \end{aligned}$$

After integrating along t from 0 to ∞, interchanging the order of integration and applying the formula for the K-Bessel function $K_{1/2}(v,w)$ given in Lemma 4.4, we arrive at the lower bound

$$\begin{aligned}\int_0^\infty HTrK_{M,\varepsilon,\delta}(t)\,dt \;&\geq\; \int_{e^\delta}^\infty \sum_{n=1}^\infty \frac{\log u}{u^n-1}\,d\left[\pi_M(u) - \sum_{0\leq\lambda_{M,j}<\varepsilon} \mathrm{li}(u^{s_{M,j}})\right] \\ &-\; \sum_{0\leq\lambda_{M,j}<\varepsilon}\left(\int_{-\infty}^0 e^{s_{M,j}\xi}\,d\xi + \int_0^\delta e^{(s_{M,j}-1)\xi}\,d\xi\right) \\ &=\; \int_{e^\delta}^\infty \sum_{n=1}^\infty \frac{\log u}{u^n-1}\,d\left[\pi_M(u) - \sum_{0\leq\lambda_{M,j}<\varepsilon} \mathrm{li}(u^{s_{M,j}})\right] \\ &-\; 1-\delta-\sum_{0<\lambda_{M,j}<\varepsilon}\left(\frac{1}{s_{M,j}} + \frac{1-e^{-(1-s_{M,j})\delta}}{1-s_{M,j}}\right).\end{aligned}$$

Proceeding now as in the proof of Theorem 4.7, we then obtain the lower bound

$$\begin{aligned}c_M \;&\geq\; 1+\sum_{0<\lambda_{M,j}<\varepsilon}\frac{1}{\lambda_{M,j}} + \sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta}} \frac{Z_\gamma'}{Z_\gamma}(1) - C_{M,\varepsilon}e^{-(1-s_\varepsilon)\delta} \\ &-\; 1-\delta-\sum_{0<\lambda_{M,j}<\varepsilon}\left(\frac{1}{s_{M,j}} + \frac{1-e^{-(1-s_{M,j})\delta}}{1-s_{M,j}}\right).\end{aligned}$$

Observing the relation

$$\frac{1}{\lambda_{M,j}} = \frac{1}{s_{M,j}} + \frac{1}{1-s_{M,j}},$$

the latter estimate for c_M simplifies to

$$c_M \geq -\delta + \sum_{0<\lambda_{M,j}<\varepsilon} \frac{e^{-(1-s_{M,j})\delta}}{1-s_{M,j}} + \sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta}} \frac{Z_\gamma'}{Z_\gamma}(1) - C_{M,\varepsilon}e^{-(1-s_\varepsilon)\delta}.$$

Choosing now $0<\varepsilon<\min(\lambda_{M,1}, 7/64)$, we obtain the lower bound

$$c_M \geq -\delta + \sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta}} \frac{Z_\gamma'}{Z_\gamma}(1) - C_{M,\varepsilon}e^{-(1-s_\varepsilon)\delta}.$$

Finally, choosing

$$\delta = \delta_{M,\varepsilon} = \max\left\{\frac{s_\varepsilon}{\varepsilon}\log(C_{M,\varepsilon}), 5\right\}$$

(observing $N_{M;0,\varepsilon} = 1$), we end up with the lower bound

$$c_M \geq -1 - \delta_{M,\varepsilon} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1) =: c_M^{\mathrm{low}}.$$

Comparing this with the upper bound

$$c_M \leq 3 + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1) =: c_M^{\mathrm{up}},$$

yields

$$c_M^{\mathrm{up}} - c_M^{\mathrm{low}} = 4 + \delta_{M,\varepsilon}.$$

4.10. Remark. Fix a Fuchsian group $\Gamma_0 \subseteq \mathrm{PSL}_2(\mathbb{R})$ of the first kind and denote the associated quotient space $\Gamma_0 \backslash \mathbb{H}$ by M_0. For subgroups $\Gamma \subseteq \Gamma_0$ of finite index in Γ_0 the associated quotient spaces $M = \Gamma \backslash \mathbb{H}$ are then finite covers of M_0 of degree $[\Gamma_0 : \Gamma]$. If C_{M_0} is a positive constant implied by the prime geodesic theorem for M_0 (cf. 2.9), it is shown in [11] that one can take

$$C_M = [\Gamma_0 : \Gamma] \cdot C_{M_0}$$

as the corresponding constant in the prime geodesic theorem for M. From this and the bound for $N_{M;0,1/4}$ given in 2.7 one immediately derives that for any $0 < \varepsilon \leq 7/64$ there exists a constant $D_{M_0,\varepsilon} > 0$ depending only on M_0 and ε such that the quantity $\delta_{M,\varepsilon}$ defined in Theorem 4.7 has the upper bound

$$\delta_{M,\varepsilon} \leq D_{M_0,\varepsilon} \cdot \log \mathrm{vol}(M).$$

4.11. Theorem. Let M be a finite cover of M_0 as in Remark 4.10. Then, we have the upper bound

$$c_M \leq \sum_{0 < \lambda_{M,j} < \varepsilon} \frac{1}{\lambda_{M,j}} + O_{M_0,\varepsilon}\left(\mathrm{vol}(M) \cdot (\log \mathrm{vol}(M))^2\right).$$

Proof. By Theorem 4.7, it suffices to prove an upper bound for

$$\sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1).$$

For this purpose, we denote by $\tilde{H}(\Gamma_0)$ a complete set of representatives of the Γ_0-conjugacy classes of the hyperbolic elements of Γ_0 and by $r_{\Gamma_0,\Gamma}$ the maximal number of fusions of different hyperbolic Γ-conjugacy classes to the same hyperbolic Γ_0-conjugacy class. Then, we define the map

$$h : H(\Gamma) \longrightarrow \tilde{H}(\Gamma_0)$$

by associating to $\gamma \in H(\Gamma)$ the element $\tilde{\gamma}_0 \in \tilde{H}(\Gamma_0)$ representing the Γ_0-conjugacy class containing γ. We note that the fiber $h^{-1}(\tilde{\gamma}_0)$ of an element $\tilde{\gamma}_0 \in \tilde{H}(\Gamma_0)$ has at most $r_{\Gamma_0,\Gamma}$ elements. From this we deduce the estimate

$$\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1) \;\leq\; r_{\Gamma_0,\Gamma} \sum_{\substack{\tilde{\gamma}_0\in \tilde{H}(\Gamma_0)\\ \ell_{\tilde{\gamma}_0}<\delta_{M,\varepsilon}}} \frac{Z'_{\tilde{\gamma}_0}}{Z_{\tilde{\gamma}_0}}(1).$$

Now, for each $\tilde{\gamma}_0 \in \tilde{H}(\Gamma_0)$ there exists uniquely $\gamma_0 \in H(\Gamma_0)$ and $n \in \mathbb{N}$ satisfying $\tilde{\gamma}_0 = \gamma_0^n$; for the hyperbolic lengths of the closed geodesics determined by $\tilde{\gamma}_0$ and γ_0 we note the equality $\ell_{\tilde{\gamma}_0} = n\ell_{\gamma_0}$. In particular, if $\tilde{\gamma}_0 \in \tilde{H}(\Gamma_0)$ with $\ell_{\tilde{\gamma}_0} < \delta_{M,\varepsilon}$ and $\tilde{\gamma}_0 = \gamma_0^n$ with $\gamma_0 \in H(\Gamma_0), n \in \mathbb{N}$ as before, we have the estimate

$$n < \frac{\delta_{M,\varepsilon}}{\ell_{\gamma_0}} < \frac{\delta_{M,\varepsilon}}{\ell_0},$$

where $\ell_0 > 0$ is a real number smaller than the length of the shortest, closed geodesic on M_0. This leads to the estimate

$$\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1) \;\leq\; r_{\Gamma_0,\Gamma} \sum_{\substack{\gamma_0\in H(\Gamma_0)\\ \ell_{\gamma_0}<\delta_{M,\varepsilon}}} \sum_{n<\delta_{M,\varepsilon}/\ell_0} \frac{Z'_{\gamma_0^n}}{Z_{\gamma_0^n}}(1).$$

Finally, using the fact that $Z'_{\gamma_0^n}/Z_{\gamma_0^n}(1)$ is monotone decreasing with respect to n, we arrive at the upper bound

$$\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1) \;\leq\; r_{\Gamma_0,\Gamma} \cdot \frac{\delta_{M,\varepsilon}}{\ell_0} \sum_{\substack{\gamma_0\in H(\Gamma_0)\\ \ell_{\gamma_0}<\delta_{M,\varepsilon}}} \frac{Z'_{\gamma_0}}{Z_{\gamma_0}}(1).$$

Now, one easily proves that there exists a constant $D_1 > 0$ depending only on M_0 such that the estimate

$$\frac{Z'_{\gamma_0}}{Z_{\gamma_0}}(1) \leq D_1 \frac{\ell_{\gamma_0}}{e^{\ell_{\gamma_0}}}$$

holds true. Furthermore, by the prime geodesic theorem, there exists a constant $D_2 > 0$ depending only on M_0 such that

$$d\pi_{M_0}(u) \leq D_2 \frac{du}{\log u}.$$

Combining the three preceding estimates we end up with

$$\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta_{M,\varepsilon}}} \frac{Z'_\gamma}{Z_\gamma}(1) \leq r_{\Gamma_0,\Gamma} \cdot \frac{\delta_{M,\varepsilon}}{\ell_0} \int\limits_{e^{\ell_0}}^{e^{\delta_{M,\varepsilon}}} D_1 \frac{\log u}{u}\, d\pi_{M_0}(u) \leq$$

$$\frac{D_1 D_2}{\ell_0} \cdot r_{\Gamma_0,\Gamma} \cdot \delta_{M,\varepsilon} \int\limits_{e^{\ell_0}}^{e^{\delta_{M,\varepsilon}}} \frac{\log u}{u} \frac{du}{\log u} = \frac{D_1 D_2}{\ell_0} \cdot r_{\Gamma_0,\Gamma} \cdot \delta_{M,\varepsilon}(\delta_{M,\varepsilon} - \ell_0).$$

Applying the estimate for $\delta_{M,\varepsilon}$ given in Remark 4.10 together with the obvious bound $r_{\Gamma_0,\Gamma} \leq [\Gamma_0 : \Gamma]$ concludes the proof of the theorem. $\square$

5 Examples: Congruence subgroups

5.1. Definitions. For a positive, natural number N, we denote by $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$ (resp.) the following congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$:

$$\begin{aligned} \Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,|\, c \equiv 0 \bmod N \right\}, \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,|\, a \equiv d \equiv 1 \bmod N,\, c \equiv 0 \bmod N \right\}, \\ \Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}. \end{aligned}$$

We let $\Gamma_0 = \mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma = \overline{\Gamma}_0(N)$, $\overline{\Gamma}_1(N)$, $\overline{\Gamma}(N)$, where $\overline{\Gamma}_0(N)$, $\overline{\Gamma}_1(N)$, $\overline{\Gamma}(N)$ denotes the image of $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$ in $\mathrm{PSL}_2(\mathbb{Z})$ (resp.). We denote by $Y_0(N)$, $Y_1(N)$, $Y(N)$ the quotient space associated to $\overline{\Gamma}_0(N)$, $\overline{\Gamma}_1(N)$, $\overline{\Gamma}(N)$ (resp.). Since the discussions for the lower and upper bounds for $c_{Y_0(N)}$, $c_{Y_1(N)}$, $c_{Y(N)}$ are essentially identical and lead to similar results, we will focus here on the discussion for $\Gamma = \overline{\Gamma}_0(N)$ and simply state without proof the corresponding results in the other two cases. To simplify the discussion, we assume from now on that N is square-free.

5.2. Lower bounds. From [22], p. 24, we have

$$[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}_0(N)] = \prod_p (p+1) = O(N \log N).$$

Therefore, the hyperbolic volume of $Y_0(N)$ becomes

$$\mathrm{vol}(Y_0(N)) = 2\pi \cdot \frac{[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}_0(N)]}{6} = O(N \log N).$$

Furthermore, with $\omega(N)$ denoting the number of primes dividing N and taking into account the asymptotic $\omega(N) \sim \log N / \log\log N$ as N tends to infinity (through specific sequences of square-free numbers; cf. [5], p. 355), we derive from [22], p. 25, that the total branching order $b_{Y_0(N)}$ of $Y_0(N)$ can be bounded by

$$b_{Y_0(N)} \leq 3 \cdot 2^{\omega(N)} = O\left(N^{\log 2/\log\log N}\right) = O_\varepsilon(N^\varepsilon)$$

for any given $\varepsilon > 0$; for the number $p_{Y_0(N)}$ of cusps of $Y_0(N)$ we deduce from [22], p. 25, the rough estimate

$$p_{Y_0(N)} = O([\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}_0(N)] = O(N \log N).$$

By means of the above estimates, we then deduce from Theorem 3.3 the lower bound

$$c_{Y_0(N)} \geq -4\log\left(\frac{[\mathrm{PSL}_2(\mathbb{Z}):\overline{\Gamma}_0(N)]}{6} + 2p_{Y_0(N)} + 5b_{Y_0(N)} + 2N_{Y_0(N);0,1/4}\right),$$

which is obviously seen to be of the order $O(\log N)$ with an implied constant roughly equal to -4.

In the same way one can prove that $c_{Y_1(N)}$, resp. $c_{Y(N)}$ also have lower bounds of the order $O(\log N)$ with an implied constant roughly equal to -8, resp. -12.

5.3. Upper bounds. Considering $Y_0(N)$ as a cover over the fixed base space $Y_0(1)$ we derive by means of Theorem 4.11 the trivial upper bound

$$c_{Y_0(N)} \leq O_\varepsilon\left(N^{1+\varepsilon}\right)$$

for any given $\varepsilon > 0$. We now show how this upper bound can be improved using results of M. Huxley (cf. [10]) on the number of conjugacy classes of hyperbolic matrices of bounded trace in $\Gamma_0(N)$. For this we start with the upper bound given in Corollary 4.8, namely

$$c_{Y_0(N)} \leq 3 + \sum_{\substack{\gamma\in H(\Gamma_0(N))\\ \ell_\gamma<\delta_{Y_0(N)}}} \frac{Z'_\gamma}{Z_\gamma}(1) \leq 3 + 4\cdot \sum_{\substack{\gamma\in H(\Gamma_0(N))\\ \ell_\gamma<\delta_{Y_0(N)}}} \frac{\ell_\gamma}{e^{\ell_\gamma}};$$

for the second inequality we have taken into account that $|\mathrm{tr}(\gamma)| \geq 3$. Furthermore, we derive from Remark 4.10 that there is a positive, even integer k_0 independent of N such that

$$\delta_{Y_0(N)} \leq \log(N^{k_0}).$$

Now, for $\gamma \in H(\Gamma_0(N))$, we put $t := |\mathrm{tr}(\gamma)|$ and recall

$$3 \leq t = 2\cosh(\ell_\gamma/2) = e^{\ell_\gamma/2} + e^{-\ell_\gamma/2},$$

from which we immediately derive the inequalities

$$\ell_\gamma \leq 2\log t\,,\ \ \frac{1}{e^{\ell_\gamma}} \leq \frac{4}{t^2}.$$

In particular, we observe that, if γ runs through the set $H(\Gamma_0(N))$ with $\ell_\gamma < \delta_{Y_0(N)}$, then t ranges across integers between 3 and $2N^{k_0/2}$. Denoting by $\nu(t)$ the number of $\Gamma_0(N)$-conjugacy classes of matrices of trace t and by $f(t)$ the function $\log t/t^2$, we arrive at the estimate

$$c_{Y_0(N)} \leq 3 + 2\cdot 4\cdot \sum_{t=3}^{2N^{k_0/2}} \nu(t)\cdot 2\log t\cdot \frac{4}{t^2} \leq 3 + 64\cdot \sum_{t=3}^{2N^{k_0/2}} \nu(t)f(t).$$

Setting $T := 2N^{k_0/2}$ and $F(T) := \sum_{t=3}^{T} \nu(t)$, we then compute using summation by parts

$$
\begin{aligned}
c_{Y_0(N)} &\leq 3 + 64 \left(F(T)f(T) + \sum_{t=3}^{T-1} F(t)(f(t) - f(t+1)) \right) \\
&\leq 3 + 64 \left(F(T)f(T) + \sum_{t=3}^{T-1} F(t)|f'(t)| \right),
\end{aligned}
$$

since $f(t)$ is a convex function for $t \geq 3$. By [10], p. 66, we now have the estimate

$$
F(T) = O_\varepsilon \left(N^\varepsilon T^{2+\varepsilon} \right),
$$

which leads to the upper bound

$$
c_{Y_0(N)} \leq O_\varepsilon \left(N^\varepsilon T^\varepsilon + \sum_{t=3}^{T-1} N^\varepsilon t^{-1+\varepsilon} \right) = O_\varepsilon \left(N^\varepsilon T^\varepsilon \right).
$$

Since $T = 2N^{k_0/2}$, we finally obtain the desired upper bound

$$
c_{Y_0(N)} \leq O_\varepsilon \left(N^\varepsilon \right)
$$

for any given $\varepsilon > 0$.

Since the analogue of the function $F(T)$ for the groups $\Gamma_1(N)$, resp. $\Gamma(N)$ has the same upper bound as $F(T)$ (cf. [10], p. 66), we also obtain for $c_{Y_1(N)}$, resp. $c_{Y(N)}$ upper bounds of the form $O_\varepsilon \left(N^\varepsilon \right)$.

Acknowledgement: Both authors would like to thank A. Abbes, Y. André, J.-M. Bismut, J.-B. Bost, S. Kudla, K. Murty, C. Soulé, E. Ullmo, G. Wüstholz for many inspiring discussions. In particular, the authors would like to thank P. Sarnak for pointing out an error in the original proof of the upper bound for $c_{Y_0(N)}$ in the last section.

References

[1] *Abbes, A.; Ullmo, E.:* Auto-intersection du dualisant relatif des courbes modulaires $X_0(N)$. J. Reine Angew. Math. **484** (1997), 1-70.

[2] *Burger, M.:* Small eigenvalues of Riemann surfaces and graphs. Math. Z. **205** (1990), 395-420.

[3] *Buser, P.:* Geometry and spectra of compact Riemann surfaces. Progress in Math. **106**. Birkhäuser-Verlag, Boston-Basel-Berlin, 1992.

[4] *Chavel, J.:* Eigenvalues in Riemannian geometry. Pure Appl. Math. **115**. Academic Press, New York, 1984.

[5] *Hardy, G.; Wright, E.:* An introduction to the theory of numbers, fifth edition. The Clarendon Press, Oxford University Press, New York, 1979.

[6] *Hejhal, D.:* The Selberg trace formula for $\mathrm{PSL}_2(\mathbb{R})$, vol. 1. Lecture Notes in Math. **548**. Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[7] *Hejhal, D.:* The Selberg trace formula for $\mathrm{PSL}_2(\mathbb{R})$, vol. 2. Lecture Notes in Math. **1001**. Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[8] *Hejhal, D.:* A classical approach to a well-known spectral correspondence on quaternion groups. In: Number Theory, New York, 1983-84, 127-196. Chudnovsky, D.V.; Chudnovsky, G.V.; Cohn, H.; Nathanson, M.B. (eds.). Lecture Notes in Math. **1135**. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.

[9] *Huntley, J.; Jorgenson, J.; Lundelius, R.:* On the asymptotic behaviour of counting functions associated to degenerating hyperbolic Riemann surfaces. J. Funct. Anal. **149** (1997), 58-82.

[10] *Huxley, M.:* Conjugacy classes in congruence subgroups. In: Proceedings of the Conference on Automorphic Forms and Analytic Number Theory, 65-88. Murty, R. (ed.). Publications CRM. Université de Montréal, Montréal, 1990.

[11] *Jorgenson, J.; Kramer, J.:* On the error term of the prime geodesic theorem. Preprint 2000.

[12] *Jorgenson, J.; Kramer, J.:* Bounds on Faltings's delta function through covers. Preprint 2000.

[13] *Jorgenson, J.; Lang, S.:* Basic Analysis of Regularized Products and series. Lecture Notes in Math. **1564**. Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona-Budapest, 1993.

[14] *Jorgenson, J.; Lang, S.:* Explicit formulas. Lecture Notes in Math. **1593**. Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona-Budapest, 1994.

[15] *Jorgenson, J.; Lundelius, R.:* A regularized heat trace for hyperbolic Riemann surfaces of finite volume. Comment. Math. Helv. **72** (1997), 636-659.

[16] *Kühn, U.:* Über die arithmetischen Selbstschnittzahlen zu Modulkurven und Hilbertschen Modulflächen. Dissertation, Humboldt-Universität, Berlin, 1999.

[17] *Kühn, U.:* Generalized arithmetic intersection numbers. C. R. Acad. Sci. Paris **327** (1998), 283-288.

[18] *Lundelius, R.:* Asymptotics of the determinant of the Laplacian on hyperbolic surfaces of finite volume. Duke Math. J. **71** (1993), 211-242.

[19] *McKean, H.:* Selberg's trace formula as applied to a compact Riemann surface. Comm. Pure Appl. Math. **25** (1972), 225-246. Correction: Comm. Pure Appl. Math. **27** (1974), 134.

[20] *Michel, P.; Ullmo, E.:* Points de petite hauteur sur les courbes $X_0(N)$. Invent. Math. **131** (1998), 645-674.

[21] *Selberg, A.:* Remarks on the distribution of poles of Eisenstein series. Collected papers, vol. II, 15-45. Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[22] *Shimura, G.:* Arithmetic theory of automorphic functions. Princeton University Press, Princeton, New Jersey, 1971.

[23] *Zograf, P.G.:* Fuchsian groups and small eigenvalues of the Laplacian operator. Soviet Math. Dokl. **26** (1984), 1618-1621.

Jay Jorgenson
Department of Mathematics
City College of New York
Convent Avenue at 138th Street
New York, NY 10031
U.S.A.
e-mail: jjorgen@math0.sci.ccny.cuny.edu

Jürg Kramer
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin
Germany
e-mail: kramer@mathematik.hu-berlin.de