Bounding the sup-norm of automorphic forms

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1 Introduction

1.1. In the theory of Riemann surfaces there are a number of natural metrics which arise from various points of view. From the setting of uniformization theory, every Riemann surface has a unique, up to scale, metric which has constant curvature and is compatible with its complex structure. On the other hand, in the setting of algebraic geometry, every compact Riemann surface can be embedded via the Abel-Jacobi map into its Jacobian variety. The Jacobian admits a canonically defined flat metric which can be pulled back to the Riemann surface, yielding the canonical metric. Since the canonical metric can be expressed in terms of a basis of holomorphic 1-forms, we extend the notion of a canonical metric to any finite volume, hyperbolic Riemann surface on which there exist non-zero cusp forms of weight two. In this article, we consider an arbitrary Riemann surface which admits hyperbolic and canonical metrics, and we obtain an optimal bound relating the two metrics. More precisely, we bound the canonical metric form in terms of the hyperbolic metric form for any finite volume Riemann surface M which is a finite degree covering of a fixed Riemann surface M_0.

1.2. Let $M = \Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ denotes the upper half-plane and Γ is a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$, and assume (the compactification of) M has genus $g_M > 0$. Let $\{f_1, \ldots, f_{g_M}\}$ be an orthonormal basis of cusp forms of weight 2 with respect to Γ, normalized to have L^2-norm equal to one. If M is a finite degree cover of the hyperbolic Riemann surface $M_0 = \Gamma_0\backslash\mathbb{H}$, then our main result is the sup-norm bound

$$\sup_{z\in\mathbb{H}} \sum_{j=1}^{g_M} \mathrm{Im}(z)^2 |f_j(z)|^2 = O_{M_0}(1). \tag{1}$$

As we will see, this result can be interpreted as saying that the ratio $g_M \cdot \mu_M^{\mathrm{can}}/\mu_M^{\mathrm{hyp}}$, where μ_M^{can} is the canonical metric form of M (in some references also called the Arakelov metric form of M) and μ_M^{hyp} is the hyperbolic metric form of M, is uniformly bounded through covers. Here the canonical metric form μ_M^{can} is normalized in the standard way such that M obtains canonical volume $\mathrm{vol}_M^{\mathrm{can}}$ equal to one. On the other hand, the hyperbolic metric form μ_M^{hyp} is scaled in such a way that the hyperbolic volumes $\mathrm{vol}_M^{\mathrm{hyp}}$ of M and $\mathrm{vol}_{M_0}^{\mathrm{hyp}}$ of M_0 are related by the formula

$$\mathrm{vol}_M^{\mathrm{hyp}} = [\Gamma_0 : \Gamma] \cdot \mathrm{vol}_{M_0}^{\mathrm{hyp}} .$$

We point out that the scaling of μ_M^{can} could also be taken in such a way that the canonical volume $\mathrm{vol}_M^{\mathrm{can}}$ equals g_M, so that both volumes would have the same order of growth. We prefer to work with the standard normalization giving M canonical volume equal to one.

1.3. Our motivation for the present paper comes from [1] and [13]. There, the authors undertake detailed investigations concerning the sup-norm bound (1) in the special case of automorphic forms associated to the congruence subgroups $\Gamma_0(N)$, for square-free N which is prime to 6; their strongest result is obtained in [13], Theorem 1.5, where P. Michel and E. Ullmo prove the estimate

$$\sup_{z\in\mathbb{H}} \sum_{j=1}^{g_M} \mathrm{Im}(z)^2 |f_j(z)|^2 = O\left(\tau^5(N)\log(N)\right)$$

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with $\tau(N)$ denoting the number of divisors of N and $M = \Gamma_0(N)\backslash\mathbb{H}$. Thereby, they improve the bound $O_\varepsilon(N^{2+\varepsilon})$ which is the main result in [1], Théorème B (which, using results of J. Hoffstein and P. Lockhart [8], can be improved to $O_\varepsilon(N^{1+\varepsilon})$). Our main theorem (1) applies in this specific case, thus improving the bound obtained in [13] from $O_\varepsilon(N^\varepsilon)$ to $O_{M_0}(1)$ with $M_0 = \Gamma_0(1)\backslash\mathbb{H}$.
In general, by comparing the volumes of any surface under consideration relative to the hyperbolic metric and to the canonical metric, one can easily see that our bound is optimal.
In summary, our main theorem proves an optimal sup-norm bound for the average value of any orthonormal basis of weight two automorphic forms. Our bound goes beyond the above cited references in that we consider general towers of discrete subgroups that act on the hyperbolic upper half-plane $\mathbb{H}$ (not just the congruence subgroups $\Gamma_0(N)$); in addition, our technique applies equally well to finite volume quotients of arbitrary symmetric spaces and domains of holomorphy, going beyond the restriction of compactness as in [3].

1.4. Our proof follows two steps. First, we use local analysis and the maximum principle for subharmonic functions to study the canonical metric form near the cusps and the elliptic fixed points. Second, we use long time asymptotics of heat kernels on forms to prove bounds away from the cusps and the elliptic fixed points. The idea of using asymptotics for heat kernels on forms in order to relate the canonical or, more generally, the Bergman metric form to the hyperbolic metric form comes from [3]. Since the maximum principle holds quite generally, our main theorem stated here indeed does hold in the setting considered in [3]. For the sake of brevity, we will address the specific problem associated to quotients of the hyperbolic upper half-plane and simply state that no obstruction exists when adapting the necessary notation in order to apply the proof of our main result here to the setting of general quotients of domains of holomorphy.
In the forthcoming article [11], we again study the canonical metric, this time obtaining a precise relation between the canonical metric form and the hyperbolic metric form, expressing the difference in terms of the hyperbolic heat kernel. The techniques we employ in [11] are based on analytic methods from the Arakelov theory of algebraic curves. Connections with other analytic quantities are being investigated, such as special values of classical Eisenstein series, hyperbolic Eisenstein series as defined in [12], as well as their analogues for elliptic subgroups. The strength of the present article is the elementary proof of the optimal sup-norm bound (1), which extends and improves on results obtained in the papers [1], [3], [13] and [15]. We view the present article as a completion to the problem studied in these references as well as a complement to the in depth analysis developed in [11].

2 Background material

2.1. Hyperbolic and canonical metrics. Let Γ be a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy\}$. We let M be the quotient space $\Gamma\backslash\mathbb{H}$ and denote by g_M the genus of M. Let S be the set of elliptic fixed points and cusps, so S is a subset of points on $\overline{M}$, the closure of M obtained by (formally) adding one point for each cusp of M. Let S_∞ be the subset of S consisting of the cusps, so $S_\infty = \overline{M} \setminus M$.
We denote by μ_M^{hyp} the $(1,1)$-form associated to the hyperbolic metric of M, which is compatible with the complex structure of M and has constant negative curvature equal to minus one. Locally, we have

$$\mu_M^{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{y^2}\,,$$

where by abuse of notation, here and henceforth, we identify M locally with its universal cover $\mathbb{H}$. Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f, g\rangle = \int_M f(z)\,\overline{g(z)}\,y^k\,\frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}\,;$$

here $f, g \in S_k(\Gamma)$. By choosing an orthonormal basis $\{f_1, ..., f_{g_M}\}$ of $S_2(\Gamma)$ with respect to the Petersson inner product, we define the canonical metric form μ_M^{can} of M by

$$\mu_M^{\mathrm{can}}(z) := \frac{1}{g_M} \cdot \frac{i}{2} \sum_{j=1}^{g_M} |f_j(z)|^2 \, \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

We note that the canonical metric measures the volume of M to be one. With this notation, the function we study in the present article is

$$\frac{g_M \cdot \mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)} = \sum_{j=1}^{g_M} \mathrm{Im}(z)^2 |f_j(z)|^2.$$

2.2. A differential operator. Define by $\mathcal{S}_1(\Gamma)$ the $\mathbb{C}$-vector space of functions g on $\mathbb{H}$, for which the integral

$$\int\limits_{\Gamma\backslash\mathbb{H}} |g(z)|^2 \, \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} \,,$$

is bounded, and which satisfy the transformation formula

$$g\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{c\bar{z}+d}\right) \cdot g(z) \tag{2}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. From [5], p. 144, with $k = 2$, or [7], p. 338, with $m = 2$ (in both references the multiplier system v, as defined in [7], p. 332, is trivial) we recall the differential operator

$$\mathbf{D}_1 := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2iy\frac{\partial}{\partial x} \,,$$

which acts on the vector space $\mathcal{S}_1(\Gamma)$. By results of [7] (namely: the definition on the bottom of p. 382; proposition 5.14 (a), (d) on p. 383; the first line of the proof on p. 383) it can be shown that a C^2-function $g \in \mathcal{S}_1(\Gamma)$ lies in the kernel of $\mathbf{D}_1$, if and only if $g(z)/y$ is holomorphic, i.e., $g(z)/y$ is a cusp form of weight 2 with respect to Γ.

2.3. Heat kernels. Let $K_{\mathbb{H}}^{(1)}(t; z, w)$ $(t \in \mathbb{R}_{>0};\, z, w \in \mathbb{H})$, resp. $K_M^{(1)}(t; z, w)$ $(t \in \mathbb{R}_{>0};\, z, w \in M)$ denote the heat kernel on $\mathbb{H}$, resp. M associated to the differential operator $\mathbf{D}_1$. ¿From [5], p. 157 (see also [2], p. 539) one has explicit expressions for these heat kernels, which we now quote: For $t > 0$ and $r > 0$, let

$$g_1(t; r) := \Lambda_{1,0}(r) + \frac{e^{-t/4}}{(4\pi t)^{3/2}(\cosh(r) - 1)} \int\limits_r^\infty \frac{u e^{-u^2/4t}}{\sqrt{2}\sqrt{\cosh(u) - \cosh(r)}} G(u; r) \, \mathrm{d}u \,,$$

where

$$\Lambda_{1,0}(r) := \frac{1}{2\pi(\cosh(r) + 1)} \,,$$

and

$$\begin{aligned} G(u; r) \;&:=\; e^{-2\theta} \left(\sqrt{2}\sinh(u/2) + \sqrt{\cosh(u) - \cosh(r)}\right)^2 \\ &+\; e^{2\theta} \left(\sqrt{2}\sinh(u/2) - \sqrt{\cosh(u) - \cosh(r)}\right)^2 \end{aligned}$$

with $e^{\pm\theta}$ defined by

$$e^{\pm\theta} \sinh(r) := e^u - \cosh(r) \pm e^{u/2}\sqrt{2(\cosh(u) - \cosh(r))} \,.$$

We note that $\lim_{r\to 0} g_1(t;r)$ exists and is given by

$$g_1(t;0) := \lim_{r\to 0} g_1(t;r) = \frac{1}{4\pi} + \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_0^\infty \frac{2u\cosh(u)e^{-u^2/4t}}{\sqrt{2}\sqrt{\cosh(u)-1}}\,\mathrm{d}u\,.$$

The heat kernel on $\mathbb{H}$ associated to $\mathbf{D}_1$ is then given by

$$K_{\mathbb{H}}^{(1)}(t;z,w) := g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,w))\,,$$

where $\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(\cdot,\cdot)$ denotes the hyperbolic distance on $\mathbb{H}$. The function $\Lambda_{1,0}$ is defined in [5], p. 152 (see also [2], p. 539) in terms of the classical hypergeometric function.

Direct calculations show that $g_1(t;r)$ is positive for all $t>0$ and $r\geq 0$, and that the equality

$$\lim_{t\to\infty} g_1(t;0) = \frac{1}{4\pi}$$

holds. The heat kernel on M associated to $\mathbf{D}_1$ is given by the Poincaré series (see [5], p. 157, or [2], p. 540)

$$K_M^{(1)}(t;z,w) := \sum_{\gamma\in\Gamma} \left(\frac{c\bar{w}+d}{cw+d}\right)\left(\frac{z-\gamma(\bar{w})}{\gamma(w)-\bar{z}}\right) g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,\gamma(w)))\,,$$

where

$$\gamma(w) = \frac{aw+b}{cw+d} \text{ with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

note that in the above definition we have again identified M and $\mathbb{H}$ locally. We also recall from [5], p. 154, that the series defining $K_M^{(1)}(t;z,w)$ converges absolutely and uniformly on compacta for fixed $t\geq\delta$, for any $\delta>0$, and (z,w) ranging over $M\times M$, away from a sufficiently small neighborhood of the diagonal. Note however, since $g_1(t;r)$ is bounded and continuous for $r=0$ and $t\geq\delta$, we conclude that $K_M^{(1)}(t;z,w)$ also converges absolutely and uniformly on compacta for fixed $t\geq\delta$ and (z,w) ranging in a neighborhood of the diagonal of $M\times M$.

2.4. Spectral expansions. The heat kernel $K_M^{(1)}(t;z,w)$ admits a spectral expansion, which we now quote from [7], specifically proposition 5.3 on p. 317, and the subsequent discussion on p. 414. Let $\{g_{\lambda_j}\}_{\lambda_j\geq 0}$ be an orthonormal basis of L^2-eigenfunctions of $\mathbf{D}_1$ when acting on $\mathcal{S}_1(\Gamma)$, where g_{λ_j} has eigenvalue λ_j. For each cusp P, let $E_P(z;s)$ be the appropriately defined Eisenstein series corresponding to P (again, we refer to [7] for details). Then, the heat kernel $K_M^{(1)}(t;z,w)$ admits the spectral expansion

$$K_M^{(1)}(t;z,w) = \sum_{\lambda_j\geq 0} g_{\lambda_j}(z)\,\overline{g_{\lambda_j}(w)}\,e^{-\lambda_j t} + \sum_{P\in S_\infty} \frac{1}{2\pi}\int\limits_0^\infty e^{-(1/4+r^2)t}E_P(z;1/2+ir)\overline{E_P(w;1/2+ir)}\mathrm{d}r\,,$$

where the first sum is taken over all eigenvalues $\lambda_j\geq 0$ (counting multiplicities) and the second sum is over the set S_∞ of inequivalent cusps P of M. As proved in [7], p. 414, the above series converges absolutely and uniformly on compacta for $t>0$ and $(z,w)\in\mathbb{H}\times\mathbb{H}$.

For our purposes, there are three main points concerning the heat kernel $K_M^{(1)}(t;z,w)$ which play an important role in our work. First, the function $K_M^{(1)}(t;z,z)$ is a well-defined, Γ-invariant function on M; this claim is easily verified using the group theoretic expansion of $K_M^{(1)}(t;z,z)$, using that the factor of automorphy defining the space $\mathcal{S}_1(\Gamma)$ (cf. formula (2)) is a complex number of modulus one. Second, since all the eigenvalues λ_j are non-negative, it is immediate from the spectral expansion that $K_M^{(1)}(t;z,z)$ is a monotone decreasing function in t for fixed $z\in M$. Finally, we observe

$$\lim_{t\to\infty} K_M^{(1)}(t;z,z) = \sum_{\lambda_j=0} |g_{\lambda_j}(z)|^2 = y^2\sum_{j=1}^{g_M} |f_j(z)|^2\,,$$

where $\{f_1,...,f_{g_M}\}$ is an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. In particular, we note the formula

$$g_M\cdot\mu_M^{\mathrm{can}}(z) = \lim_{t\to\infty} K_M^{(1)}(t;z,z)\cdot\mu_M^{\mathrm{hyp}}(z).$$

3 The main result

3.1. Theorem. *Let $\Gamma_0 \subseteq \mathrm{PSL}_2(\mathbb{R})$ be a fixed Fuchsian subgroup of the first kind and $\Gamma \subseteq \Gamma_0$ any subgroup of finite index, i.e., $M = \Gamma\backslash\mathbb{H}$ is a finite cover of the fixed base space $M_0 = \Gamma_0\backslash\mathbb{H}$. Then, we have the estimate*

$$g_M \cdot \mu_M^{\mathrm{can}} = O_{M_0}\left(\mu_M^{\mathrm{hyp}}\right);$$

in terms of functions, this bound can be expressed as saying

$$\sup_{z\in\mathbb{H}} \sum_{j=1}^{g_M} y^2|f_j(z)|^2 = O_{M_0}(1)\,.$$

Proof. We first investigate the canonical metric form μ_M^{can} in neighborhoods of the cusps and the elliptic fixed points, reducing the problem to that of bounding μ_M^{can} on compacta on M, which is easily verified using heat kernel techniques.

Pick any cusp P of M and, as usual, uniformize M so that P is mapped to $i\infty$, hence, with the local coordinate $z = x + iy$ on the upper half-plane $\mathbb{H}$, we have locally around P

$$\frac{g_M \cdot \mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)} = \sum_{j=1}^{g_M} y^2|f_j(z)|^2.$$

For fixed $\varepsilon > 0$, let $B(\varepsilon)$ denote the neighborhood of P on M of area ε corresponding to the strip

$$\mathcal{S}_{a,T}(\varepsilon) = \{z \in \mathbb{H} \,|\, \mathrm{Im}(z) > T, 0 \leq \mathrm{Re}(z) < a\}\,,$$

where a is the cusp width of P with respect to Γ and, by an elementary volume computation, $T = a/\varepsilon$. Using the trivial inequality

$$\frac{y}{a} \leq \frac{1}{2\pi e} e^{2\pi y/a} = \frac{1}{2\pi e}\left|e^{-2\pi i z/a}\right|,$$

we have for $\mathrm{Im}(z) > T$ the bound

$$\sum_{j=1}^{g_M} y^2|f_j(z)|^2 \leq \left(\frac{a}{2\pi e}\right)^2 \sum_{j=1}^{g_M}\left|\frac{f_j(z)}{e^{2\pi i z/a}}\right|^2.$$

Since f_j is a cusp form, the function $f_j(z)/e^{2\pi i z/a}$ is bounded and holomorphic on the closed set

$$\overline{\mathcal{S}}_{a,T}(\varepsilon) = \{z \in \mathbb{H} \,|\, \mathrm{Im}(z) \geq T, 0 \leq \mathrm{Re}(z) < a\} \cup \{i\infty\}\,,$$

which is isometric to a closed disc. Hence, the function $\left|f_j(z)/e^{2\pi i z/a}\right|$ is subharmonic on $\overline{S}_{a,T}(\varepsilon)$ (see [16], p. 362). The strong maximum principle for subharmonic functions now applies, both to the individual functions $\left|f_j(z)/e^{2\pi i z/a}\right|$ as well as the sum of these functions (see [6], p. 15). The maximum principle states that the maximum of each function $\left|f_j(z)/e^{2\pi i z/a}\right|$ on $\overline{S}_{a,T}(\varepsilon)$ occurs on its boundary

$$\partial\overline{\mathcal{S}}_{a,T}(\varepsilon) = \{z \in \mathbb{H} \,|\, \mathrm{Im}(z) = T, 0 \leq \mathrm{Re}(z) < a\}\,.$$

Therefore, we get

$$\sup_{\mathrm{Im}(z)\geq T}\left(\sum_{j=1}^{g_M}\left|\frac{f_j(z)}{e^{2\pi i z/a}}\right|^2\right) = \sup_{\mathrm{Im}(z)=T}\left(\sum_{j=1}^{g_M}\left|\frac{f_j(z)}{e^{2\pi i z/a}}\right|^2\right).$$

Combining with the above inequalities, we then have

$$\sup_{\mathrm{Im}(z)\geq T}\left(\sum_{j=1}^{g_M} y^2|f_j(z)|^2\right) \leq \left(\frac{a}{2\pi e}\right)^2 \sup_{\mathrm{Im}(z)=T}\left(\sum_{j=1}^{g_M}\left|\frac{f_j(z)}{e^{2\pi i z/a}}\right|^2\right).$$

Noting that $y = T$ on the boundary $\partial\overline{\mathcal{S}}_{a,T}(\varepsilon)$, we arrive at

$$\sup_{z\in\overline{B}(\varepsilon)}\left(\frac{g_M\cdot\mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)}\right) \leq \left(\frac{e^{2\pi T/a}}{2\pi e T/a}\right)^2 \sup_{\mathrm{Im}(z)=T}\left(\sum_{j=1}^{g_M} y^2|f_j(z)|^2\right),$$

which on M can be rewritten as

$$\sup_{z\in\overline{B}(\varepsilon)}\left(\frac{g_M\cdot\mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)}\right) \leq \left(\frac{e^{2\pi/\varepsilon}}{2\pi e/\varepsilon}\right)^2 \sup_{z\in\partial\overline{B}(\varepsilon)}\left(\frac{g_M\cdot\mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)}\right).$$

At this time, we have focused our attention on the neighborhood $B(\varepsilon)$ of a cusp. Our argument carries through with only slight notational changes when considering a neighborhood of an elliptic fixed point; we do not present these details. Rather, we now can assert safely that we have shown the existence of a constant $C(\varepsilon)$, independent of M, depending solely on ε such that

$$\sup_{z\in M}\left(\frac{g_M\cdot\mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)}\right) \leq C(\varepsilon) \sup_{z\in M\setminus\bigcup B(\varepsilon)}\left(\frac{g_M\cdot\mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)}\right),$$

where the union is taken over all cusps and elliptic fixed points of M. Trivially, ε can be taken depending solely on M_0. Thus, in order to prove the asserted claim, it suffices to prove

$$\sup_{z\in M\setminus\bigcup B(\varepsilon)}\left(\frac{g_M\cdot\mu_M^{\mathrm{can}}(z)}{\mu_M^{\mathrm{hyp}}(z)}\right) = O_{M_0}(1).$$

We now give the proof of this assertion, which is a simple adaptation of an argument from [3] and [10]. By 2.4, we have

$$g_M\cdot\mu_M^{\mathrm{can}}(z) = \lim_{t\to\infty} K_M^{(1)}(t;z,z)\cdot\mu_M^{\mathrm{hyp}}(z).$$

Since the function $K_M^{(1)}(t;z,z)$ is monotone decreasing in t for fixed $z\in M$, we obtain the estimate

$$g_M\cdot\mu_M^{\mathrm{can}}(z) \leq K_M^{(1)}(t;z,z)\cdot\mu_M^{\mathrm{hyp}}(z) \leq \sum_{\gamma\in\Gamma} g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,\gamma(z)))\cdot\mu_M^{\mathrm{hyp}}(z)$$

for any $t\geq\delta>0$. Noting that $\Gamma_0\supseteq\Gamma$, and since $g_1(t;r)$ is positive, we immediately obtain the upper bound

$$g_M\cdot\mu_M^{\mathrm{can}}(z) \leq \sum_{\gamma\in\Gamma_0} g_1(t;\mathrm{d}_{\mathbb{H}}^{\mathrm{hyp}}(z,\gamma(z)))\cdot\mu_M^{\mathrm{hyp}}(z).$$

One can choose ε sufficiently small, depending solely on M_0, so that the neighborhoods $B(\varepsilon)$ on M are lifts of neighborhoods of the cusps and the elliptic fixed points of M_0 with properties as utilized above. With this choice of ε, and because of the uniform boundedness of the right-hand side for fixed $t\geq\delta>0$ and $z\in M_0$ on compacta (see section 2.3), the claim follows. □

3.2. Corollary. Let f_{H} be a newform for $\Gamma_0(N)$, which is Hecke-normalized, i.e., the first Fourier coefficient in its q-expansion is one. Then, for any $\varepsilon>0$, we have the bound

$$\sup_{z\in\mathbb{H}}|yf_{\mathrm{H}}(z)| = O_\varepsilon\left(N^{1/2+\varepsilon}\right).$$

Proof. Let $c_f>0$ be such that $c_f f_{\mathrm{H}}$ has L^2-norm equal to one, so then $c_f f_{\mathrm{H}}$ can be chosen to be an element in an orthonormal basis of $S_2(\Gamma_0(N))$. By Theorem 3.1 (with $\Gamma_0=\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma=\Gamma_0(N)$), we then have

$$\sup_{z\in\mathbb{H}}|yf_{\mathrm{H}}(z)| = O(c_f^{-1}).$$

The main results of [8] and [9] provide bounds for c_f in terms of N; however, note that the L^2-eigenfunctions in [8] differ from those here by a factor of the form $\mathrm{vol}_{\mathrm{hyp}}(\Gamma_0(N)\backslash\mathbb{H})^{1/2}$ because of the difference in normalizations of the hyperbolic measure (compare our section 2.1 to [8], p. 161). Consequently, by the results of [8] and [9], specifically the remarks from [8], pp. 163-164, we have (in our normalization)

$$c_f^{-1} = O_\varepsilon\left(N^\varepsilon \cdot \mathrm{vol}_{\mathrm{hyp}}(\Gamma_0(N)\backslash\mathbb{H})^{1/2}\right).$$

Using the well-known formula for $\mathrm{vol}_{\mathrm{hyp}}(\Gamma_0(N)\backslash\mathbb{H})$, the result follows. □

3.3. Remark. Corollary 3.2 reproves Théorème A of [1]. It seems likely that Corollary 3.2 can be improved; we speculate that the optimal bound is of the form

$$\sup_{z\in\mathbb{H}} |y f_{\mathrm{H}}(z)| = O_\varepsilon(N^\varepsilon).$$

3.4. Remark. The technique of embedding a basis of holomorphic forms into a basis of Maass forms has been used in other investigations; in specific arithmetic contexts regarding holomorphic forms of weight one, where no harmonic analysis is possible, this technique was used in [4] and [14]. The authors thank the referee for providing this interesting remark and the corresponding references.

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