Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 25th June

11.1 (2+4 points) Let B be a standard d-dimensional Brownian motion.

a) Prove the "0-1-law" for the "tail" σ-field

$$\hat{\mathcal{F}} := \bigcap_{t \geq 0} \sigma(X_s,\, s \geq t),$$

i.e. show that $P[A] \in \{0, 1\}$ for every $A \in \hat{\mathcal{F}}$.

b) Use a) to prove the *Liouville's Theorem*: All bounded harmonic functions on $\mathbb{R}^d$ are constant. (A function h is called harmonic if the Laplace-operator $\Delta h = 0$.)

11.2 (1+5 points) Let B be a standard Brownian motion on the Wiener space, and let $X_t = B_t + \alpha t$ $(t \in [0, T])$ be the corresponding Brownian motion with drift $\alpha \in \mathbb{R}$ and start in $X_0 = 0$. Show that:

a) The distribution P^α of X is equivalent to the Wiener measure P with the density function

$$\frac{dP^\alpha}{dP} = \exp\left(\alpha B_T - \frac{1}{2}\alpha^2 T\right).$$

b) The distribution of the maximum

$$M_T = \max_{0 \leq t \leq T} X_t$$

under P resp. the distribution of $\max_{0 \leq t \leq T} B_t$ under P^α is given by

$$P[M_T \leq c] = \Phi\left(\frac{c - \alpha T}{\sqrt{T}}\right) - e^{2\alpha c}\Phi\left(\frac{-c - \alpha T}{\sqrt{T}}\right), \quad c \geq 0.$$

Hint: Use the "reflection principle" for the joint distribution of B_T and $\max_{0 \leq t \leq T} B_t$ under P:

$$P[\max_{t \in [0,T]} B_t \geq m, B_T \leq m - x] = P[B_T \geq m + x] \quad \text{for} \quad m, x \geq 0.$$

11.3 (4 points) Let $A, T : [0,\infty) \to [0,\infty)$ be two continuous increasing functions. We consider A as a distribution function of a positive measure dA on $[0,\infty)$ and interprete T as time change. Prove the following formula:

$$\int_{T_0}^{T_t} f(s)dA_s = \int_0^t f(T_s)dA_{T_s}$$

for all measurable functions $f : [0,\infty) \to [0,\infty)$.

11.4 (4 extra points) Let B^x be a 3-dimensional Brownian motion starting at $0 \neq x \in \mathbb{R}^3$, i.e. $B^x = x + B$, where B is a standard 3-dimensional Brownian motion. Let further

$$M_t := \frac{1}{|B_t^x|}, \quad t \geq 0.$$

Show that

a) M solves $dM_t = M_t^2 dB_t$ and thus is a local martingale.

b) M is bounded in L^2, i.e. $\sup_{t\geq 0} E[M_t^2] < \infty$, and hence M is uniformly integrable.

c) We have $\lim_{t\to\infty} E[M_t] = 0$. In particular M is not a "real" martingale.

Problems 11.1 -11.3 should be solved at home and delivered at Wednesday, the 2nd July, before the beginning of the tutorial. Problem 11.4 is additional.