Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 23rdth April

2.1 (2 points) Let $(B^i_t, t \geq 0), i = 1, 2$ be two independent Standard Brownian motions on some probability space $(\Omega, \mathcal{F}, P)$. Prove that

$$B_t := \frac{1}{\sqrt{2}}(B^1_t + B^2_t), \qquad t \geq 0$$

is also a Standard Brownian motion.

2.2 (2+2 points) Let a be a continuous function of finite variation and let x be a continuous function with continuous quadratic variation $\langle x \rangle$ on $[0, T]$.

a) Show that $x + a$ has continuous quadratic variation and

$$\langle x + a \rangle = \langle x \rangle.$$

b) Show that for $f \in C^1[0, T]$ and $t \in [0, T]$ there exists the Itô-Integral

$$\int_0^t f(x_s + a_s) dx_s$$

and for $y := x + a$ it holds

$$\int_0^t f(x_s + a_s) dx_s = \int_0^t f(y_s) dy_s - \int_0^t f(y_s) da_s.$$

2.3 (2+2 points) Let $(B_t, t \geq 0)$ be a Standard Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$.

a) Compute

$$\int_0^t B^n_s dB_s,$$

where $n \in \mathbb{N}$.

b) Let

$$W_t := \sigma B_t + \mu t, \qquad t \geq 0$$

for some $\sigma \neq 0, \mu \in \mathbb{R}$. Show that

$$X_t := \exp\left(\sigma B_t + \mu t - \frac{\sigma^2}{2}t\right), \qquad t \geq 0$$

is a solution of a stochastic differential equation

$$dX_t = X_t dW_t, \qquad X_0 = 1$$

in the Itô-sense.

2.4 (1+2+2 points) Let $(B_t)_{t\geq 0}$ be a Standard Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$.

a) Let $h(s)$ $(0 \leq s \leq 1)$ be a continuous function of finite variation and with $h(1) = 0$. Using Itô's product formula show that for P-almost each trajectory $B(\omega)$ the equality

$$\int_0^1 h(s)dB_s(\omega) = -\int_0^1 B_s(\omega)dh(s), \tag{1}$$

holds, and conclude that

$$E\left[\int_0^1 h(s)dB_s\right] = 0.$$

b) Wiener and Paley have used (1) for *definition* of stochastic integral on the left-hand side (the right-hand side is well defined in classical measure theory). Moreover, they have extended the definition of the integral for arbitrary deterministic integrals $h \in L^2[0,1]$ using the isometry

$$E\left[\left(\int_0^1 h(s)dB_s\right)^2\right] = \int_0^1 h(s)^2 ds. \tag{2}$$

Prove (2) and sketch the construction of the "Wiener-integral" via isometry.

c) Show that the random variable $M_t := \int_0^t h(s)dB_s$ is normally distributed with expectation 0 and variance $\int_0^t (h(s))^2 ds$ for each $t \in [0,1]$.
Hint: The limit of P-a.s. convergent normally distributed random variables is again a normally distributed random variable.

The problems 2.1 -2.4 should be solved at home and delivered at Wednesday, the 30th April, before the beginning of the tutorial.