Stochastic Processes II Summer term 2008
(Stochastische Analysis)
Prof. Dr. Uwe Küchler
Dr. Irina Penner

Exercises, 7th Mai

4.1 (3 points) We denote by $h_n(x,t),\ n = 0,1,\dots$ the *Hermite-polynomials* defined by

$$\exp(\alpha x - \frac{1}{2}\alpha^2 t) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} h_n(x,t), \qquad x \in \mathbb{R},\ t \geq 0,$$

i.e.

$$h_n(x,t) = \frac{d^n}{d\alpha^n} \exp(\alpha x - \frac{1}{2}\alpha^2 t) \mid_{\alpha=0} .$$

Show that for a continuous function $(x_t)_{t\geq 0}$ with continuous quadratic variation $\langle x \rangle$ and $x_0 = 0$ it holds

$$\begin{aligned} h_n(x_t, \langle x \rangle_t) &= n \int_0^t h_{n-1}(x_s, \langle x \rangle_s) dx_s \\ &= n! \int_0^t \dots \int_0^{t_{n-1}} dx_{t_n} \dots dx_{t_1}, \qquad n = 1,2,\dots . \end{aligned}$$

In particular the functions $H_n := h_n(x, \langle x \rangle),\ n = 0,1,\dots$ solve the Itô-equation

$$dH_n = nH_{n-1}dx.$$

4.2 (4 points) Let $(X_t, t \geq 0)$ be a geometric Brownian motion, i.e.

$$X_t = x_0 \exp(\sigma B_t + \alpha t), \quad t \geq 0$$

where $x_0 > 0$ is a real number, $(B_t, t \geq 0)$ a Standard Brownian motion and $\sigma > 0, \alpha \in R_1$.

i) Calculate all moments $E[X_t^p], \quad t \geq 0, p \in R$.
ii) For which σ, α the process $X_t,\ t \geq 0)$ is a martingale?

iii) Prove that for every $n \geq 1$ the process $(h_n(B_t, t),\ t \geq 0)$ is a martingale. (Notation as in exercise 4.1)

4.3 (3 points) Let $(B_t, t \geq 0)$ be a Standard Brownian motion.

i) Show that for $\alpha, \sigma, x_0 \in R$ the solution of

$$dX_t = -aX_t dt + \sigma dB_t, \quad X_0 = x_0$$

is given by

$$\begin{aligned} X_t &= \mathrm{e}^{-at}(x_0 + \int_0^t \mathrm{e}^{as} \sigma dB_s) \\ &= \mathrm{e}^{-at} x_0 + \sigma B_t - a\sigma \int_0^t \mathrm{e}^{-a(t-s)} B_s ds,\ t \geq 0. \end{aligned}$$

ii) Calculate $E[X_t], Var[X_t], Cov[X_s, X_t]$ for $s, t \geq 0$.
$((X_t)$ is called the Ornstein-Uhlenbeck-process.)

4.4 (3 points) Consider the Brownian Bridge

$$Z_t := (1-t) \int_0^t \frac{1}{1-s}\, dB_s, \quad t \in [0,1], Z_1 := 0$$

(with $(B_t)_{0 \leq t \leq 1}$ standard BM).
Show that Z solves the Itô-equation

$$dZ_t = dB_t - \frac{Z_t}{1-t} dt$$

on $(0,1)$ with $Z_0 = Z_1 = 0$.