Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 21st May

6.1 (4 points) Let $(M_t)_{t\geq 0}$ be a non-negative continuous martingale that converges to 0 with $t \to \infty$ and let $M^* := \sup_{t\geq 0} M_t$. For $x > 0$ prove the following inequality

$$P[\,M^* \geq x \,|\, \mathcal{F}_0\,] = 1 \wedge \frac{M_0}{x}.$$

Hint: Consider $T_x := \inf\{\, t \geq 0 \,|\, M_t \geq x\}$ and apply the martingale convergence theorem to the process $(M_{T_x \wedge t})_{t\geq 0}$.

6.2 (2+2 points) Use problem 1 to determine distribution functions of the following random variables:

a) The maximum of a Brownian motion with start in $x > 0$ before the first visit in 0.

b) The maximum of a Brownian motion with start in 0 and with a negative drift $-m < 0$, i.e. $M^* := \sup_{t\geq 0}(B_t - mt)$, where (B_t) denotes a standard Brownian motion.
Hint: Consider $M_t := \exp(2m(B_t - mt))\ (t \geq 0)$.

6.3 (3+2 points) Let

$$X_t = x + \sigma B_t + mt \qquad (t \geq 0)$$

be a Brownian motion with start in $x \in \mathbb{R}$, variance $\sigma^2 > 0$ and drift $m \in \mathbb{R}$. (Here $(B_t)_{t\geq 0}$ denotes a standard Brownian motion.)

a) For $a < x < b$ compute the probability that X attains the value b before a.
Hint: For $m \neq 0$ determine $\lambda \in \mathbb{R}$ such that $(e^{\lambda X_t})_{t\geq 0}$ is a martingale.

b) If $m \geq 0$, the stopping time $T_b := \inf\{t \geq 0 \mid X_t = b\}$ is a.s. finite. Compute the Laplace-transform $E[e^{-\lambda T_b}]$ $(\lambda \geq 0)$.

6.4 (5 points) Let U be a uniformly integrable set of random variables and let $\bar{U}$ denote the closure of U in $L^1(P)$. Prove that $\bar{U}$ is also uniformly integrable.

The problems 6.1 -6.4 should be solved at home and delivered at Wednesday, the 28th May, before the beginning of the tutorial.