Stochastic Processes II Summer term 2008
(Stochastische Analysis)
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Exercises, 28th May

7.1 (4+3+1 points) For a standard Brownian motion $(B_t)_{t\geq 0}$ and $a \geq 0$ consider the first hitting time

$$T_a := \inf\left\{ t \geq 0 \,\middle|\, B_t = a \right\} = \inf\left\{ t \geq 0 \,\middle|\, B_t \geq a \right\}$$

and the first passage time

$$T_{a^+} := \inf\left\{ t \geq 0 \,\middle|\, B_t > a \right\}.$$

Show that:

a) The process $T_a \ \ (a \geq 0)$ is a left-continuous increasing process, and the process $T_{a^+} \ \ (a \geq 0)$ is its right-continuous version, i.e. $T_{a^+} = \lim_{b\downarrow a} T_b$ and $P[\,T_a = T_{a^+}\,] = 1$ for all $a \geq 0$.

b) For $b > a$ is $T_b - T_a$ independent of $\mathcal{A}_{T_a}$ and has the same distribution as T_{b-a}, i.e. it the process $T_a \ \ (a \geq 0)$ has stationary independent increments.

c) T_a and $a^2 T_1$ have the same distribution for all $a \geq 0$.

7.2 (3+2 points) Let $(\mathcal{A}_t)_{t\geq 0}$ be a filtration satisfying usual conditions. An adapted cadlag process $(X_t)_{t\geq 0}$ with $X_0 = \text{const}$ is called a local martingale, if there exists a "localizing sequence" of stopping times $(T_n)_{n\in\mathbb{N}}$, such that $T_n \to \infty$ P-a.s., and the stopped process $(X_{t\wedge T_n})_{t\geq 0}$ is a martingale with respect to $(\mathcal{A}_t)_{t\geq 0}$ for all $n \in \mathbb{N}$. Let $(X_t)_{t\geq 0}$ be a local martingale. Show that:

a) If (X_t) is continuous, then the sequence of stopping times

$$S_n := \inf\left\{ t \geq 0 \,\middle|\, |X_t| > n \right\} \qquad (n \in \mathbb{N})$$

is a localizing sequence for (X_t), and also any other sequence of stopping times $(R_n)_{n\in\mathbb{N}}$ with $R_n \leq S_n$ and $R_n \to \infty$ P-a.s. Thus we can assume without loss of generality that $(X_{t\wedge T_n})_{t\geq 0}$ is bounded for all $n \in \mathbb{N}$.

b) If $X_t \geq 0$ for all $t \geq 0$, then $(X_t)_{t\geq 0}$ is a supermartingale.

7.3 (3+2 points)

a) Let $(M_t)_{t\geq 0}$ be a continuous process such that both $(M_t, \mathcal{A}_t, t \geq 0)$ and $(M_t^2, \mathcal{A}_t, t \geq 0)$ are local martingales. Prove that the process (M_t) is P-a.s. constant.

b) Let $(M_t, \mathcal{A}_t, t \geq 0)$ be a continuous local martingale with paths of finite variation. Prove that the process (M_t) is P-a.s. constant.

The problems 7.1 -7.3 should be solved at home and delivered at Wednesday, the 4th June, before the beginning of the tutorial.