Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 30th January

14.1 (4 points) Consider the following game: A fair coin (each side appears with probability $\frac{1}{2}$) is tossed repeatedly, the tosses are independent of each other. Before each toss a gambler with the initial fortune $X_0 > 0$ stakes the $(1-a)$th fraction of his current fortune on tail ($a \in (0,1)$ is fixed). If the toss shows tail, the gambler recives the double of his stake back. Otherwise he loses his stake. Let X_n, $n = 0, 1, \ldots$ denote the fortune of the gambler after the nth toss.

a) Show that (X_n) is a martingale, i.e. the game is fair.
b) Prove that $\lim_n X_n = 0$ P-a.s., i.e. the gambler will be a.s. ruined.

14.2 (5 points) Let X_n be the amount of water in a reservoir at noon at day n. During the 24 hour period beginning at this time, a quantity Y_n of water flows into the reservoir, and just before noon on each day exactly one unit of water is removed (if this amount can be found). The maximum capacity of the reservoir is K, and excessive inflows are spilled and lost. Assume that Y_n are independent and identically distributed random variables and that, by rounding off to some laughably small unit of volume, all numbers in this exercise are non-negative integers.

a) Show that (X_n) is a Markov chain and determine its transition matrix.
b) Find an expression for the invariant distribution in terms of the generating function G of the Y_n.
c) Find the invariant distribution when Y_1 has generating function $G(s) = p(1-(1-p)s)^{-1}$, $p \in (0,1)$.

14.3 (3 points) Show by example that Markov chains which are not irreducible may have many different invariant distributions.

14.4 (4 points) Recall that the Poisson process is defined as

$$N(t) = \sum_{k=1}^{\infty} \mathbb{1}_{\{S_k \le t\}}, \quad t \ge 0, \quad \text{with} \quad S_k = \tau_1 + \cdots + \tau_k,$$

where τ_k ($k = 1, 2, \ldots$) are independent identically exponentially distributed random variables with parameter $\lambda > 0$ on some probability space $(\Omega, \mathcal{A}, P)$. Show that the Poisson process is continuous in probability and continuous P-a.s. at each point $t \in (0, \infty)$ but it does not have continuous trajectories.

The problems 14.1 -14.4. should be solved at home and delivered at Wednesday, the 6th February, before the beginning of the tutorial.