Exercises, 21st November

6.1 (4 points) Assume that some box contains b black and w white balls ($b, w \geq 1$). The content of the box is changed as follows: One ball is drawn randomly, its color is denoted, and this ball is placed back into the box together with c other balls of the same color ($c \in \mathbb{N}_0$ fixed). This procedure is repeated infinitely often. We denote by X_n the number of the black and by Y_n the number of the white balls in the box after the n-th turn ($n \in \mathbb{N}_0$). Let further $\mathfrak{A}_n = \sigma(X_0, X_1, \ldots, X_n)$.
Show that the proportions

$$V_n := \frac{X_n}{X_n + Y_n}\ , \quad n = 0, 1, \ldots$$

of the black balls in the box form a martingale w.r.t. $(\mathfrak{A}_n)_{n\in\mathbb{N}_0}$. Can V_n be represented as a sum of mutually independent random variables?

6.2 (2 points) Let X_n ($n \in \mathbb{N}$) be a sequence of i. i. d. random variables on some probability space $(\Omega, \mathfrak{A}, P)$. We define

$$S_n := \sum_{k=1}^{n} X_k, \quad \text{and} \quad \mathfrak{A}_n := \sigma(X_1, X_2, \ldots, X_n), \qquad n = 1, 2, \ldots.$$

Assume further that $Z(\lambda) := E[\mathrm{e}^{\lambda X_1}] < \infty$ for some $\lambda \in \mathbb{R}$ and consider

$$M_n := \frac{\exp[\lambda S_n]}{Z(\lambda)^n}, \quad n = 1, 2, \ldots.$$

Prove that the sequence $(M_n, \mathfrak{A}_n)_{n\in\mathbb{N}}$ is a martingale.

6.3 a) (2 points) Let $(B_t)_{t\geq 0}$ be a stochastic process on some probability space $(\Omega, \mathfrak{A}, P)$. Show that the following conditions are equivalent:

i) $(B_t)_{t\geq 0}$ is a standard Brownian motion.

ii) $B_0 = 0$ and for all $n \in \mathbb{N}$ and all $0 < t_1 < t_2 < \ldots < t_n < \infty$ the vector $(B_{t_1}, \ldots, B_{t_n})$ has a Gaussian distribution with the expectation 0 and the covariance matrix $\Sigma = (t_i \wedge t_j)_{1 \leq i,j \leq n}$. (That means $(B_t)_{t \geq 0}$ is a Gaussian process.)

b) (3 points) Let $(B_t)_{t \geq 0}$ be a standard Brownian motion on some probability space $(\Omega, \mathfrak{A}, P)$. Show that the following stochastic process are also standard Brownian motions:

i) For $s \geq 0$:

$$\tilde{B}_t := B_{t+s} - B_s, \quad t \geq 0.$$

ii) For $c \in \mathbb{R} \setminus 0$:

$$\tilde{B}_t := cB_{\frac{t}{c^2}}, \quad t \geq 0.$$

("Scaling property")

iii)

$$\tilde{B}_t := -B_t, \quad t \geq 0.$$

("Reflection")

6.4 (4 points) Assume that $S = (\tau_k, k \in \mathbb{N})$ is a sequence of independent identically exponentially distributed random variables with parameter $\lambda > 0$. Let further

$$\sigma_n := \sum_{k=1}^{n} \tau_k, \quad n \geq 1, \quad \sigma_0 := 0,$$

and consider the Poisson process

$$N_t = \sum_{k=1}^{\infty} \mathbb{1}_{[0,t]}(\sigma_k), \quad t \geq 0$$

and the sequence $S_n := (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $n \geq 1$. Prove that

$$P(S_n \in B | N_t = n) = \frac{n!}{t^n} \int_B \mathbb{1}_{\triangle_t^{(n)}}(t_1, t_2, \ldots, t_n) dt_1 \ldots dt_n$$

holds for every $n \geq 1, t > 0$ and for all $B \in \mathcal{B}_n$. Here we use the notation

$$\triangle_t^{(n)} = \{(t_1, t_2, \ldots, t_n) \in R_n : 0 < t_1 < \ldots < t_n < t\}.$$

The problems 6.1 -6.4. should be solved at home and delivered at Wednesday, the 28th November, before the beginning of the tutorial.