Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 5th December

8.1 (5 points) Let τ be a random variable on $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{N}_0$. We define

$$X_n := \mathbb{1}_{[0,n]}(\tau) \quad \text{and} \quad \mathcal{A}_n := \sigma(X_0, \ldots, X_n), \quad n \in \mathbb{N}_0.$$

Then $(X_n, \mathcal{A}_n)_{n\in\mathbb{N}_0}$ is a submartingale due to the lecture. We denote by A_n $(n \in \mathbb{N}_0)$ the predictable increasing process from the Doob decomposition of (X_n). Show that

$$A_n = \sum_{k=1}^{\tau\wedge n} P[\tau = k | \tau > k-1], \qquad n \in \mathbb{N}_0.$$

8.2 Let X_n, $n \in \mathbb{N}$, be i.i.d. random variables on $(\Omega, \mathcal{A}, P)$ and let $\mathcal{A}_n := \sigma(X_1, \ldots, X_n), n \in \mathbb{N}_0$. We define

$$S_0 := 0, \quad S_n := X_1 + \ldots + X_n, \quad n \in \mathbb{N}.$$

a) (2 points) Show that if $E[X_1] = 0$ and $\sigma^2 := \mathrm{var}[X_1] < \infty$, then the process
$$M_n := S_n^2 - n\sigma^2, \quad n = 0, 1, \ldots,$$
is a martingale with respect to $(\mathcal{A}_n)_{n=0,1,\ldots}$.

b) (2 points) Let $(S_n)_{n\in\mathbb{N}_0}$ be a random walk with $p = \frac{1}{2}$, i.e.
$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2}. \end{cases}$$
Then $M_n = S_n^2 - n\sigma^2$, $n = 0, 1, \ldots$ is a martingale due to part a). For $a \in \{0, 1, \ldots\}$ we define the stopping time
$$T := \min\{n \geq 0 | \, |S_n| = a\}.$$
Use the stopping theorem to compute the expected value $E[T]$.

8.3 Let $(S_n, \mathcal{A}_n)_{n\in\mathbb{N}_0}$ be a random walk with $p = \frac{1}{2}$ as in problem 8.2 b).

a) (2 points) For a given $\lambda \geq 0$ determine a value $\alpha \in \mathbb{R}$ such that the process

$$M_n^\lambda := \exp(\alpha S_n - \lambda n), \quad n = 0, 1, \ldots,$$

is a martingale.

b) (2 points) Use $(M_n^\lambda)_{n\in\mathbb{N}_0}$ to compute for $a \in \mathbb{N}$ the Laplace-transform $E[e^{-\lambda T}]$ of the stopping time $T := \min\{n \geq 0 | |S_n| = a\}$.

8.4 (4 points) We denote by V_n the capital of some insurance company at the end of the year n, $n = 0, 1, \ldots$ and assume that $V_0 > 0$. During the year n the company receives premium c and has to pay out a random value Y_n for insured losses, i.e.

$$V_n = V_{n-1} + c - Y_n, \quad n = 1, 2 \ldots .$$

We assume that Y_n, $n = 1, 2, \ldots$ are i.i.d random variables with expected value $m \in (0, c)$ and define $Z(\lambda) := E[e^{\lambda Y_1}]$. Let R be the event "Ruin of the insurance company", i.e. $R := \{T < \infty\}$ with

$$T := \min\{n \geq 0 | V_n < 0\}.$$

Show that if λ_0 is a nontrivial solution of $Z(\lambda) = e^{\lambda c}$, then we have $\lambda_0 > 0$ and the probability of ruin satisfies the inequality

$$P[R] \leq e^{-\lambda_0 V_0}.$$

The problems 8.1 -8.4. should be solved at home and delivered at Wednesday, the 12th December, before the beginning of the tutorial.