Stochastic Processes I Winter term 2007/2008
(Stochastik II)
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Exercises, 12th December

9.1 Let $(\Omega, \mathcal{A}, (\mathcal{A}_n)_{n\in\mathbb{N}_0}, P)$ be a filtered probability space.

a) (2 points) Let $(X_n)_{n\in\mathbb{N}_0}$ be an adapted integrable process on $(\Omega, \mathcal{A}, (\mathcal{A}_n)_{n\in\mathbb{N}_0}, P)$. Show that there exists a unique decomposition

$$X_n = M_n + A_n, \qquad n = 0, 1, \ldots,$$

such that the process (M_n) is a martingale and the process (A_n) is predictable with $A_0 = 0$. This decomposition is called the *Doob-decomposition* of the process (X_n).

b) (2 points) Let $M = (M_n)_{n\in\mathbb{N}_0}$ and $N = (N_n)_{n\in\mathbb{N}_0}$ be two martingales w.r.t. $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ such that $E[M_n^2] < \infty$ and $E[N_n^2] < \infty$ for all $n \in \mathbb{N}_0$. Compute the Doob-decomposition of the process $MN = (M_n N_n)_{n\in\mathbb{N}_0}$ and describe the predictable process (A_n) in terms of the conditional covariances

$$E[(M_{n+1} - M_n)(N_{n+1} - N_n) \,|\, \mathcal{A}_n] \qquad n = 0, 1, \ldots.$$

9.2 Assume that $\Omega = [0, 1)$ and let P be the Lebesgue measure on Ω. We denote by $\mathcal{A}_n$, $n = 0, 1, \ldots$, the σ-algebra generated by the dyadic intervals

$$I_{n,k} := [k2^{-n}, (k+1)2^{-n}), \qquad k = 0, \ldots, 2^n - 1.$$

Let further F be some increasing function on $[0, 1]$.

a) (2 points) Show that

$$f_n(\omega) := \sum_{k=0}^{2^n-1} \tfrac{F((k+1)2^{-n}) - F(k2^{-n})}{2^{-n}} \, \mathbb{1}_{I_{n,k}}(\omega)$$

defines a non-negative martingale with respect to $(\mathcal{A}_n)$. Thus there exists for P-almost all $\omega \in [0, 1)$ the limit $f(\omega) := \lim_{n\to\infty} f_n(\omega)$. This limit can be seen as the “dyadic derivative” of F at ω.

b) (2 points) Assume in addition that F is Lipschitz-continuous, i.e. there exists a constant $C > 0$ such that the inequality

$$|F(\omega) - F(\widetilde{\omega})| < C|\omega - \widetilde{\omega}|$$

holds for all $\omega, \widetilde{\omega} \in \Omega$. Show that then for $0 \le a < b < 1$

$$F(b) - F(a) = \int_a^b f(\omega)\, d\omega.$$

9.3 (5 points) Let $(\mathcal{A}_n)_{n=0,-1,\dots}$ be a filtration in $(\Omega, \mathcal{A}, P)$ and let

$$\mathcal{A}_{-\infty} := \bigcap_{n=0,-1,\dots} \mathcal{A}_n.$$

Show that each supermartingale $(X_n)_{n=0,-1,\dots}$ with respect to the filtration $(\mathcal{A}_n)_{n=0,-1,\dots}$ converges P-almost sure with $n \to -\infty$ to some $\mathcal{A}_{-\infty}$-measurable random variable $X_{-\infty}$ with values in $(-\infty, +\infty]$ P-a.s..
Hint: Prove a version of an "Upcrossing lemma" for supermartingales and argue as in the proof of the convergence theorem for submartingales in the lecture.

9.4 (5 points) The following result of Blackwell, Dubins and Hunt combines the Lebesgue's dominated convergence theorem and the convergence theorem for uniformly integrable martingales.
Assume that $(X_n)_{n=0,1,\dots}$ is a sequence of random variables on $(\Omega, \mathcal{A}, P)$ with

$$X_n \to X_\infty \quad P\text{-a. s.}, \qquad \sup_n |X_n| \in \mathcal{L}^1(P),$$

let $(\mathcal{A}_n)_{n=0,1,\dots}$ be a filtration in $(\Omega, \mathcal{A}, P)$ and let $\mathcal{A}_\infty := \sigma\left(\cup_n \mathcal{A}_n\right)$. Prove that

$$\lim_{n\uparrow\infty} E[X_n \,|\, \mathcal{A}_n] = E[X_\infty \,|\, \mathcal{A}_\infty] \qquad P\text{-a.s.}$$

and even the stronger assertion

$$\lim_{n,m\uparrow\infty} E[X_n \,|\, \mathcal{A}_m] = E[X_\infty \,|\, \mathcal{A}_\infty] \quad P\text{-a.s.}$$

holds.

The problems 9.1 -9.4. should be solved at home and delivered at Wednesday, the 19th December, before the beginning of the tutorial.