Construction of a diffeomorphism of $\mathbb{CP}^1$ and $S^2$

November 17, 2006

- $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus (0,0))/\sim$ with $(z_1, z_2) \sim (ze^{i\theta}, ze^{-i\theta}) \Leftrightarrow \exists z \in \mathbb{C}^*$, s. t. $(z_1, z_2) = (zz_1, zz_2)$.

An atlas is given by $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ where the charts are

\[
\varphi_1 : \mathbb{CP}^1 \setminus [0 : 1] = U_1 \to \mathbb{C} \quad [1 : z_2] \mapsto z_2
\]

\[
\varphi_2 : \mathbb{CP}^1 \setminus [1 : 0] = U_2 \to \mathbb{C} \quad [z_1 : 1] \mapsto z_1.
\]

With $\varphi_1(U_1 \cap U_2) = \mathbb{C}^*$, the transition mapping is

\[
\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* \quad z \mapsto z^{-1}
\]

since $\varphi_2 \circ \varphi_1^{-1}(z) = \varphi_2([1 : z]) = \varphi_2([z^{-1} : 1]) = z^{-1}$ for all $z \in \mathbb{C}^*$. Here we use charts in $\mathbb{C}$ rather than in $\mathbb{R}^2$ for convenience.

- $S^2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$

An atlas is given by $\{(V_1, \psi_1), (V_2, \psi_2)\}$ where the charts are

\[
\psi_1 : S^2 \setminus (0, 0, 1) = V_1 \to \mathbb{C} \quad (a, b, c) \mapsto \frac{a}{1-c} + i \frac{b}{1-c}
\]

\[
\psi_2 : S^2 \setminus (0, 0, -1) = V_2 \to \mathbb{C} \quad (a, b, c) \mapsto \frac{a}{1+c} - i \frac{b}{1+c}
\]

With $\psi_1(V_1 \cap V_2) = \mathbb{C}^*$, the transition mapping also turns out to be

\[
\psi_2 \circ \psi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* \quad z \mapsto z^{-1}
\]

In order to see this note that for all $(a, b, c) \in S^2 \setminus (0, 0, \pm 1)$ we have

\[
\left(\frac{a}{1-c} + i \frac{b}{1-c}\right)\left(\frac{a}{1+c} - i \frac{b}{1+c}\right) = 1
\]

and apply the fact that $\psi_1$ and $\psi_2$ are both bijections, which is well-known from the construction of the stereographic projection.
• Define a map $f : \mathbb{CP}^1 \to S^2$ by setting $f(\varphi_i^{-1}(z)) := \psi_i^{-1}(z)$ for all $z \in \mathbb{C}$.

1. This is a well-defined mapping from $\mathbb{CP}^1$ to $S^2$ since

$$\varphi_2^{-1}(z_2) = \varphi_1^{-1}(z_1) \iff z_2 = \varphi_2 \circ \varphi_1^{-1}(z_1) \iff z_2 = z_1^{-1} \iff z_2 = \psi_2 \circ \psi_1^{-1}(z_1) \iff \psi_2^{-1}(z_2) = \psi_1^{-1}(z_1).$$

2. This is a smooth mapping since

$$\psi_i \circ f \circ \varphi_i^{-1} : \mathbb{C} \to \mathbb{C} \quad z \mapsto z$$

and for $i \neq j$

$$\psi_j \circ f \circ \varphi_i^{-1} : \mathbb{C}^* \to \mathbb{C}^* \quad z \mapsto z^{-1}$$

are all smooth.

3. The map is a diffeomorphism since

$$f^{-1} : S^2 \to \mathbb{CP}^1 \quad \psi_i^{-1}(z) \mapsto \varphi_i^{-1}(z) \quad \forall z \in \mathbb{C}$$

defines a smooth inverse for $f$. 