Holonomy on S^2

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(1) The parallel transport along a geodesic is easily determined for surfaces. Indeed, the tangent vector to the geodesic is mapped to a tangent vector of the geodesic, by the differential equation for geodesics. Since the transport preserves angle (to the geodesic) and length of the tangent vector to the surface this information is sufficient to determine the parallel transport along any curve which is piecewise geodesic. Now on S^2 the great circles are the geodesics (Homework Problems 11, Problem 2). Last thing to notice: three quarter circles form a triangle of geodesics which meet at right angles at the corners! Pick a orthonormal basis at a corner point consisting of tangent vectors to the two geodesics. let us follow the parallel transport in counterclockwise direction. Lets call the vector tangent to the corresponding geodesic v_1 and the other w_1. At the next corner point under parallel transport v_1 will be mapped to v_2, w_1 to w_2. v_2 will be orthogonal to the next geodesic, w_2 will be tangent to it. At the third corner v_3 will be tangent to the next geodesic to follow along and w_3 will be orthogonal to it. $V-4$ and w_4 will be the tangent vectors at the original corner obtained by parallel transport along the triangle. Draw the picture! You will see, that each vector v_4 and w_4 is obtained by rotation by 90 degrees from v_1 and w_1. Since the parallel transport is a linear map, the parallel transport along the triangle is rotation by 90 degrees.

(2) Choose angular coordinates $(\phi, \psi) \in \mathbb{R}^2 \mapsto (\cos(\phi \cos\psi, \cos\phi \sin\psi, \sin\phi) \in S^2$ such that for $\phi = const.$ these parametrize the circle in question. We compute the Christoffel symbols with respect to these coordinates:

$$\Gamma^{\phi}_{\psi\psi} = \cos\phi \sin\phi \qquad \Gamma^{\psi}_{\psi\phi} = \Gamma^{\psi}_{\phi\psi} = -\tan\phi$$

and all other coefficients vanish. Now a parallel vector field along the circle can be written as

$$X(\psi) = a(\psi)\frac{\partial}{\partial\phi} + b(\psi)\frac{\partial}{\partial\psi},$$

and will satisfy

$$0 = \nabla_{\frac{\partial}{\partial\psi}} X = a' \frac{\partial}{\partial\phi} + a \tan\phi \frac{\partial}{\partial\psi} \tag{1}$$
$$+ b' \frac{\partial}{\partial\psi} - b \cos\phi \sin\phi \frac{\partial}{\partial\phi}. \tag{2}$$

This leads to differential equations

$$a' = -b \cos\phi \sin\phi \tag{3}$$
$$b' = a \tan\phi. \tag{4}$$

Notice that $a = a(\psi)$ and $b = b(\psi)$ so this is a linear differential equation with **constant** coefficients! Solutions are given by

$$a(\psi) = \mu \cos(\psi \sin\phi) + \lambda \sin(\psi \sin\phi) \tag{5}$$
$$b(\psi) = \mu \sin(\psi \sin\phi)/ \cos\phi - \lambda \cos(\psi \sin\phi)/ \cos\phi. \tag{6}$$

Remark: First of all the geodesic is the grand circle, given by $\phi \equiv 0$. hence the formula recovers what we have said about parallel transport along geodesics. It appears as if the parallel transport around small circles about $\phi \equiv \pm\pi/2$ is a rotation by almost 360 degrees. However, notice that the (ϕ, ψ)-coordinates are not defined in these two points. With respect to coordinates around these points (e.g. stereographic projection) one observes that the basis vectors $\{\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\psi}\}$ themselves rotate along the small circle completing the full 360 degrees. Subtracting this one sees that the parallel transport along $\phi \equiv const.$ is given by rotation by $\psi((sin\phi) - 1)$.