Boundary of a differentiable manifold

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Let $\phi : U \subset \mathbb{H}^n \to M$ be a differentiable parametrization. $\mathbb{H} := \{x \in \mathbb{R}^n \mid x = (x_1, x_2, ..., x_n), x_i \in \mathbb{R}, x_1 \geq 1\}$, $\partial H := \{x \in \mathbb{R}^n \mid x_1 = 0\}$. We have seen in class, that for any other differentiable parametrization $\psi : V \subset \mathbb{H} \to M$ we have for $p \in \phi(U) \cap \psi(V)$: $p \in \phi(\partial\mathbb{H} \cap U)$ if and only if $p \in \psi(\partial\mathbb{H} \cap V)$ (Corollary of Proposition 1.15). Hence ∂M is well-defined by: $\phi^{-1}(\partial M) \subset \partial\mathbb{H}$ for any diferentiable parametrization ϕ. The homework consists of proving the following two claims:

1. $\{(\phi|_{\partial\mathbb{H}}, U \cap \partial\mathbb{H}) \mid (U, \phi), U \subset \mathbb{H}, \phi$ smooth parametrization of $M\}$ defines an atlas of a differentiable manifold on ∂M whose topology coincides with the subspace topology of $\partial M \subset M$.
2. Suppose M is oriented. If we restrict ourselves to oriented parametrizations (U, ϕ) in the above definition we obtain an atlas for ∂M which defines an orientation: we identify $\partial\mathbb{H} \cong \mathbb{R}^{n-1}$ via $(0, x_2, ..., x_n) \mapsto (x_2, x_3, ..., x_n)$ (which is a homeomorphism) and need to show that the determinant of the differentials of the transition maps is always positive (Proposition 1.15 (2)). We show that a basis of $T_p(\partial M)$ is oriented with respect to this atlas if and only if the basis of T_pM obtained by completing the former basis by putting an outward pointing vector in the first position.

(1) To see that the above defines an atlas on ∂M is simple: Let (ϕ, U) and (ψ, V) be two parametrizations of M. Since $\phi^{-1}(\phi(U) \cap \psi(V)) \subset \mathbb{H}$ is an open set by definition, its intersection with ∂H is also open. Set $\Phi := \phi|_{\partial\mathbb{H}}$ and $\Psi := \psi|_{\partial\mathbb{H}}$ and $\partial U := U \cap \partial\mathbb{H}$, $\partial V := V \cap \partial\mathbb{H}$. Consider the last two sets as open sets in $\mathbb{R}^{n-1}$ via the above identification. Then

$$\Phi^{-1}(\Phi(\partial U) \cap \Psi(\partial V)) = \partial\mathbb{H} \cap \phi^{-1}(\phi(U) \cap \psi(V)).$$

But the latter is open in $\partial\mathbb{H}$ in its relative topology which coincides with the topology of $\mathbb{R}^{n-1}$ since the identifivcation is a homeomorphism. Since $\Psi^{-1} \circ \Phi$ on this set is the restricion of the differentiable map $\psi^{-1} \circ \phi$ it is itself differentiable. The same is true for $\Phi^{-1} \circ \Psi$ (interchange the roles of ϕ and ψ), hence it is a diffeomorphism. We finally need to show that a set in ∂M which is open with respect to the relative topology of ∂M as a subset of M is also open with respect to the topology defined by the atlas just constructed and vice versa. For this it is actually enough to show that $\Phi : \partial U \to \partial M$ is a homeomorphism onto its image. We already know that this ist true for $\phi : U \to M$. I claim that the assertion follows, since Φ is the restriction of this map (Homework Problems 7, Problem 1). Indeed the restriction of a homeomorphism is a homeomorphism onto its image (Why? We conider relative topology on the domain and the image of the restriction!).

(2) For the first assertion we need to show that $\det(d_x(\Psi^{-1}\Phi)) > 0$ for all $\Phi, |Psi$ as constructed above. We will conclude this from $\det(d_x(\psi^{-1}\phi)) > 0$: Notice that the size of the matrices differ by 1. We have

$$d_x(\psi^{-1}\phi) = \begin{pmatrix} \partial(\psi^{-1}\phi)/\partial x_1 & 0\dots0 \\ \vdots & d_x(\Psi^{-1}\Phi) \end{pmatrix}.$$

Now $\partial(\psi^{-1}\phi)/\partial x_1 > 0$ (Why?) and the first claim follows. For any basis B of T_pM denote by B_ϕ the matrix of coefficients with respect to the coordinate tangent vectors of its elements for the parametrization ϕ. By definition, B is oriented if and only if $\det(B_\phi) > 0$ for any parametrization ϕ of a neighbourhood of p. Now pick a basis $\mathbf{B}$ of $T_p(\partial M)$. Assume its oriented with repect to the oriented atlas just constructed. Pick an outward pointing vector $v \in T_pM$ and set $B := \{v, \mathbf{B}\}$. Pick a parametrization ϕ of a neighbourhood of p in M from the oriented atlas. We have $v = a\partial/\partial x_1 + \dots$ for some $a > 0$ with respect to that parametrization. Then

$$B_\phi = \begin{pmatrix} a & \dots \\ 0 & \mathbf{B}_\Phi \end{pmatrix}.$$

Since $\det(\mathbf{B}_\Phi) > 0$ and $a > 0$ we have $\det(B\phi) > 0$ and B is oriented. Now assume that B is oriented for an outward pointing vector v. Then $\det(B_\phi) > 0$ since ϕ is from the oriented atlas and hence $\det(\mathbf{B}_\Phi) > 0$ since $a > 0$.