Homework Problems 1
Analysis and Geometry on Manifolds WS 06/07
due 2.11.2006

Problem 1
Let \( f : U \to \mathbb{R}^n \) be a smooth function on an open subset \( U \) with \( d_x f \neq 0 \) at some \( x \in U \). Construct coordinates \((y_1, ..., y_n)\) about \( x \) such that \( f(y_1, ..., y_n) = y_1 \).

Problem 2
(a) Consider the set of (real) lines in \( \mathbb{C}^{n+1} \),
\[
\mathbb{CP}^n := \mathbb{C}^{n+1} \setminus \{0\} / \sim
\]
where \( x \sim y \) if and only if \( x = \lambda y \) for some \( \lambda \in \mathbb{C} \). \( \mathbb{CP}^n \) is called the \( n \)-dimensional complex projective space. It is equipped with the quotient topology induced by the topology of \( \mathbb{C}^{n+1} \) and the relation. Show that this is a manifold. Hint: Construct coordinates on \( U_k := \{[x] \in \mathbb{CP}^n \mid x_k \neq 0\} \) for \( k = 1, ..., n+1 \) and show that the transformation maps are diffeomorphisms.
Show that \( \mathbb{CP}^n \) is Hausdorff.
(b) Show that \( \mathbb{CP}^1 \cong S^2 \) are diffeomorphic manifolds.

Problem 3
Show that the following conditions are equivalent:
(i) \( X \) is Hausdorff.
(ii) \[
\{x\} = \bigcap_{x \in U \subset A, U \setminus A \text{ open}} A
\]
(iii) \( \Delta := \{(x, x) \mid x \in X\} \subset X \times X \) is closed.

The following problems will be discussed in the tutorials:

Problem 4
(a) Express the Laplace operator on differentiable functions \( f = f(x_1, x_2) \) on (open subsets of) \( \mathbb{R}^2 \),
\[
\Delta f := - \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) f,
\]
in terms of polar coordinates \( (r, \theta) \mapsto (r \cos \theta, r \sin \theta) \). (b) How do bilinear forms on \( T_x \mathbb{R}^n \cong \mathbb{R}^n \) transform under coordinate changes?
(c) How does the Hessian of a function \( f \) on \( \mathbb{R}^n \),
\[
\Hess f = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j},
\]
transform under change of coordinates? What is the normal form of the Hessian at a point?
Problem 5
Show that the following topological spaces are differentiable manifolds
(a) The Klein bottle: $K^2 := [0,1] \times [0,1]/\sim$ where $\sim$ is the equivalence defined by the following relations: $(0,s) \sim (1,1-s)$ and $(t,0) \sim (t,1)$
(b) the set of un-ordered $n$-tuples of complex numbers $R := \mathbb{C}^n/S_n$, where the symmetric group acts via $(z_1, \ldots, z_n) \mapsto (z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ for a given permutation $\sigma \in S_n$. Is the same true for un-ordered $n$-tuples of real numbers?
Show that both are Hausdorff.