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Homework Problems 2

Analysis and Geometry on Manifolds WS 06/07

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The following 3 *problems are your homework assignment.*

Problem 1
Two differential structures on a set M agree if the corresponding maximal atlases agree or, equivalently, the identity map, $id_M : M \to M, \quad id_M(x) = x$ is a diffeomorphism. Let $\mathbb{R}$ be equipped with $\mathcal{A} := \{(\mathbb{R}, \varphi(x) = x)\}$ and $\mathcal{A}' := \{(\mathbb{R}, \psi(x) = x^3)\}$.
(a) Show that the differential structures defined by $\mathcal{A}$ and $\mathcal{A}'$ are not the same.
(b) Show that $(\mathbb{R}, \mathcal{A})$ and $(\mathbb{R}, \mathcal{A}')$ are diffeomorphic.

Problem 2
Let X and Y be topological spaces. The product topology on $X \times Y$ is the smallest topology (i.e. the topology consisting of the smallest set $\mathcal{T}_{X\times Y}$ of open sets) such that both projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous.
(a) Characterize the open sets in $X \times Y$
(b) Show that a map $f : Z \to X \times Y$ from another topological space Z is continuous if and only if $p_X \circ f$ and $p_y \circ f$ are continuous.

Problem 3
Show that the tangent space TM of a differentiable manifold M is a differentiable manifold. Show that the following maps are differentiable: $\pi : TM \to M$ (assigning the base point to a tangent vector) and $T_pM \cong \mathbb{R}^n \to TM$ embedding the tangent space at $p \in M$ into TM. Moreover, show that a map $X : M \to TM$ is differentiable if and only if $X = X(p)$ is a differentiable vector field.

The following problems will be discussed in the tutorials.

Problem 4
Describe the differential of a differentiable map $f : M \to N$ between differentiable manifolds in terms of the definition of tangent vectors as
(a) equivalence classes of curves starting at the base point
(b) derivations at the base point.
Show that your definitions of the differential are equivalent, via the (canonical) isomorphisms between the definitions of tangent spaces, to the definition of the differential in terms of coordinates given in the class.

Problem 5
Consider the following two factor spaces $X_i := \mathbb{R}^2/\sim_i$ where the equivalence relations $\sim_i$ $(i = 1, 2)$ are given by the following sets of equivalence classes:
(i) $\{(x, \tau + \tan^2 x) \mid x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)\}$, $\{-\frac{\pi}{2} + k\pi\} \times \mathbb{R}$ and $\{\frac{\pi}{2} + k\pi\} \times \mathbb{R}$ for $k \in \mathbb{Z}$ and $\tau \in \mathbb{R}$,
(ii) $\{(x, \tau + \tan x) \mid x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)\}$, $\{-\frac{\pi}{2} + k\pi\} \times \mathbb{R}$ and $\{\frac{\pi}{2} + k\pi\} \times \mathbb{R}$ for $k \in \mathbb{Z}$ and $\tau \in \mathbb{R}$
Show that both spaces are $T1$, X_2 is Hausdorff (T2) but X_1 is not.