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Homework Problems 4

Analysis and Geometry on Manifolds WS 06/07

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The following 3 problems are your homework assignment.

Problem 1
Consider the following map $\Phi : K^2 \to \mathbb{R}^4$ where $K^2 = [0,1] \times [0,1]/\sim$ is the Klein bottle (defined in the lecture):

$$\Phi([x,y]) := ((r\cos(2\pi y)+a)\cos(2\pi x), (r\cos(2\pi y)+a)\sin(2\pi x), r\sin(2\pi y)\cos(\pi x), r\sin(2\pi y)\sin(\pi x)).$$

Here $a > r > 0$ are real parameters. Show that Φ is differentiable and an injective immersion. Explain, why this implies that its image is a submanifold and Φ is a diffeomorphism onto its image.

Problem 2
(1) Let $F : \mathbb{R}^n \to \mathbb{R}^k$ be a differentiable map and let $a \in \mathbb{R}^k$ be a regular value of F. Describe the tangent space to the submanifold $M := F^{-1}(a)$ in terms of the map F.
(2) Let X, Y be differentiable vector fields on a differentiable manifold N which are tangent to a submanifold $M \subset N$ in all points of M. Show that:

(a) The Lie bracket $[X, Y]$ is also tangent to M in all points of M

(b) Any flow line of X as above starting at a point in M lies completely in M.

(3) Show that the differentiable vector field on $\mathbb{R}^3$ is tangent to the unit sphere S^2:

$$X(x,y,z) := xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + (z^2-1)\frac{\partial}{\partial z}$$

Explain why the flow exists for all times and compute it. Determine the long-time behaviour of the flow Φ_t for X, i.e.

$$\lim_{t\to\pm\infty} \Phi_t$$

Problem 3
Do not use the statement of Satz 1.9. for the following two problems since they are part of the proof of this statement.
(1)Let $f : M \to N$ be a bijective immersion of differentiable manifolds of the same dimension. Show that f is a diffeomorphism.
(2) Let $f : M \to N$ be an injective immersion without the assumption on the dimension from (1). Suppose that $f(M) \subset N$ is a submanifold. Show that f is an embedding of topological spaces. (*Hint: You may make use of the part (1) of this problem).*

Bitte wenden...

The following problem will be discussed in the tutorials.

Problem 4

Let $O(n) := \{A \in M(n, \mathbb{R}) \mid A^T A = \mathbb{E}\}$ *be the set of orthogonal matrices.*

(1) Show that $O(n)$ *is compact, differentiable submanifold.*

(2) Show that it is a group, where the group operation is given by the matrix multiplication. Show that $(A, B) \in O(n) \times O(n) \mapsto AB$ *and* $A \in O(n) \mapsto A^{-1}$ *are differentiable maps. Explain, why for any* $A \in O(n)$ *the map* $L_A : X \in O(n) \mapsto AX \in O(n)$ *is a diffeomorphism.*

(3) Show that the tangent space $T_{\mathbb{E}}O(n) \cong \underline{o}(n) := \{X \in M(n, \mathbb{R}) \mid X^T + X = 0\}$*. Verify that for* $X, Y \in \underline{o}(n)$ *the commutator* $[[X, Y]] := XY - YX$ *given by matrix multiplication defines an element in* $\underline{o}(n)$*.*

(4) Verify that by $\tilde{X}(A) := d_{\mathbb{E}}L_A(X)$ *for a given* $X \in \underline{o}(n)$ *we obtain a differentiable vector field, the* fundamental vector field corresponding to X. *Show that*

$$\widetilde{[[X, Y]]} = [\tilde{X}, \tilde{Y}],$$

where the bracket on the right hand side denotes the Lie bracket on vector fields.

(5)* *For a matrix* $X \in M(n, \mathbb{R})$ *denote by* $\exp(X) := \sum_{i=0}^{\infty} \frac{X^i}{i!}$*. Show that* $\exp(X) \in O(n)$ *for any* $X \in \underline{o}(n)$ *and* $\exp : \underline{o}(N) \to O(n)$ *is a diffeomorphism onto its image. Show that for* $X \in \underline{o}(n)$ *the flow* Φ_t *generated by* $\tilde{X}$ *is given by* $L_{\exp(tX)}$*.*