The following 3 problems are your homework assignment.

**Problem 1**
Consider the following map $\Phi : K^2 \to \mathbb{R}^4$ where $K^2 = [0,1] \times [0,1]/\sim$ is the Klein bottle (defined in the lecture):

$$\Phi(x, y) := ((r \cos(2\pi y) + a) \cos(2\pi x), (r \cos(2\pi y) + a) \sin(2\pi x), r \sin(2\pi y) \cos(\pi x), r \sin(2\pi y) \sin(\pi x)).$$

Here $a > r > 0$ are real parameters. Show that $\Phi$ is differentiable and an injective immersion. Explain why this implies that its image is a submanifold and $\Phi$ is a diffeomorphism onto its image.

**Problem 2**
(1) Let $F : \mathbb{R}^n \to \mathbb{R}^k$ be a differentiable map and let $a \in \mathbb{R}^k$ be a regular value of $F$. Describe the tangent space to the submanifold $M := F^{-1}(a)$ in terms of the map $F$.

(2) Let $X, Y$ be differentiable vector fields on a differentiable manifold $N$ which are tangent to a submanifold $M \subset N$ in all points of $M$. Show that:

\begin{enumerate}
  \item[(a)] The Lie bracket $[X, Y]$ is also tangent to $M$ in all points of $M$
  \item[(b)] Any flow line of $X$ as above starting at a point in $M$ lies completely in $M$.
\end{enumerate}

(3) Show that the differentiable vector field on $\mathbb{R}^3$ is tangent to the unit sphere $S^2$:

$$X(x, y, z) := xz \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} + (z^2 - 1) \frac{\partial}{\partial z}$$

Explain why the flow exists for all times and compute it. Determine the long-time behaviour of the flow $\Phi_t$ for $X$, i.e.

$$\lim_{t \to \pm \infty} \Phi_t$$

**Problem 3**
Do not use the statement of Satz 1.9. for the following two problems since they are part of the proof of this statement.

(1) Let $f : M \to N$ be a bijective immersion of differentiable manifolds of the same dimension. Show that $f$ is a diffeomorphism.

(2) Let $f : M \to N$ be an injective immersion without the assumption on the dimension from (1). Suppose that $f(M) \subset N$ is a submanifold. Show that $f$ is an embedding of topological spaces.

(\textit{Hint: You may make use of the part (1) of this problem}).

Bitte wenden...
The following problem will be discussed in the tutorials.

**Problem 4**

Let \( O(n) := \{ A \in M(n, \mathbb{R}) \mid A^T A = E \} \) be the set of orthogonal matrices.

1. Show that \( O(n) \) is compact, differentiable submanifold.

2. Show that it is a group, where the group operation is given by the matrix multiplication. Show that \((A, B) \in O(n) \times O(n) \mapsto AB \) and \( A \in O(n) \mapsto A^{-1} \) are differentiable maps. Explain why for any \( A \in O(n) \) the map \( L_A : X \in O(n) \mapsto AX \in O(n) \) is a diffeomorphism.

3. Show that the tangent space \( T_E O(n) \cong \mathfrak{o}(n) := \{ X \in M(n, \mathbb{R}) \mid X^T + X = 0 \} \). Verify that for \( X, Y \in \mathfrak{o}(n) \) the commutator \( [[X, Y]] := XY - YX \) given by matrix multiplication defines an element in \( \mathfrak{o}(n) \).

4. Verify that by \( \tilde{X}(A) := d_E L_A(X) \) for a given \( X \in \mathfrak{o}(n) \) we obtain a differentiable vector field, the fundamental vector field corresponding to \( X \). Show that

\[
[[\tilde{X}, \tilde{Y}]] = [\tilde{X}, \tilde{Y}],
\]

where the bracket on the right hand side denotes the Lie bracket on vector fields.

5. For a matrix \( X \in M(n, \mathbb{R}) \) denote by \( \exp(X) := \sum_{i=0}^{\infty} \frac{X^i}{i!} \). Show that \( \exp(X) \in O(n) \) for any \( X \in \mathfrak{o}(n) \) and \( \exp : \mathfrak{o}(N) \to O(n) \) is a diffeomorphism onto its image. Show that for \( X \in \mathfrak{o}(n) \) the flow \( \Phi_t \) generated by \( X \) is given by \( L_{\exp(tX)} \).