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Homework Problems 6

Analysis and Geometry on Manifolds WS 06/07

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Problem 1
(1) Show that the boundary ∂M of a differentiable manifold with boundary, M, is a differentiable manifold without boundary ($\partial(\partial M) = \emptyset$) with the topology induced by the topology of M. Show that the natural injection is an injective immersion. Characterize the tangent vectors in T_xM in a point $x \in \partial M$ which are tangent to ∂M (that is lie in the image of the differential of the injection).
(2) Show that the boundary of an oriented manifold with boundary is oriented. In particular, show that the description of oriented bases of $T_x(\partial M)$ in Satz 1.17 actually defines an orientation of ∂M.

Problem 2
(1) Let $n \in \mathbb{N}$ be an integer. Decide whether $\mathbb{RP}^n$ is oriented and explain your answer.
(2) Let M and N be orientable manifolds. Show that $M \times N$ is orientable. Given orientations on M and N construct a natural orientation on $M \times N$. Decide whether the map $M \times N \to N \times M$ given by $(m, n) \mapsto (n, m)$ is oriented or not with respect to these orientations.

Problem 3
Let $f : M \to \mathbb{R}$ be a differentiable manifold (without boundary) and let a be a regular value of M. Show that the *sublevel set* $\{p \in M \mid f(x) \leq a\} \subset M$ is a manifold with boundary with the topology induced by the topology of M (or empty).

The following problems will be discussed in the tutorials:

Problem 4
Exterior 2*-forms.* Show that for any given antisymmetric bilinear form $\omega \in \Lambda^2(V^*)$ on an n–dimensional real vector space V there exists a basis $\{v_1, ..., v_n\}$ of V such that (with the notation given in the lecture)

$$\omega = v^1 \wedge v^2 + v^3 \wedge v^4 + ... + v^{2r-1} \wedge v^{2r}.$$

Determine the Gram matrix with respect to this basis and its rank.

Problem 5
(1) Show that the set $\{v^{i_1} \wedge ... \wedge v^{i_k} \mid 1 \leq i_1 < i_2 < ... < i_k \leq n\}$ described in the lecture is a basis of $\Lambda^k(V^*)$. Determine the dimension $\dim(\Lambda^k(V^*))$.
(2) Denote by $T^k(V^*)$ the space of all multi-linear forms on V. Show that the map

$$A : T^k(V^*) \to \Lambda^k(V^*) \qquad A(\alpha)(w_1, ..., w_k) := \frac{1}{k!} \sum_{\sigma \in S_n} sgn(\sigma)\alpha(w_{\sigma(1)}, ..., w_{\sigma(k)})$$

is a projection onto $\Lambda^k(V^*)$, i.e. $A^2 = A$ and $A|_{\Lambda^k(V^*)} = id$.