Homework Problems 7
Analysis and Geometry on Manifolds WS 06/07
due 14.12.2006

Problem 1
Show that the following definition is equivalent to Definition 1.14 of the lecture: A manifold with boundary is a topological space $M$ which admits a family of parametrizations
\[ \{ \phi_{\alpha} : U_{\alpha} \to M \} \]
such that for all $\alpha, \beta$
(i) $U_{\alpha} \subset \mathbb{H}^n$ and $\phi_{\alpha}(U_{\alpha}) \subset M$ are open sets
(ii) $\bigcup_{\alpha} \phi_{\alpha}(U_{\alpha}) = M$
(iii) $\phi_{\alpha}$ is a homeomorphism onto its image
(iv) $\phi_{\beta}^{-1} \circ \phi_{\alpha}(\phi_{\alpha}(U_{\alpha}) \cap \phi_{\beta}(U_{\beta})) \to \phi_{\beta}^{-1}(\phi_{\alpha}(U_{\alpha}) \cap \phi_{\beta}(U_{\beta}))$ is differentiable.

Problem 2
(1) Show that the linear forms $\sigma_1, ..., \sigma_k$ on a vector space $V$ are linear independent if and only if $\sigma_1 \wedge ... \wedge \sigma_k \neq 0$.
(2) Let $\{v_1, ..., v_n\}$ be a basis of $V$ and $\{\sigma_1, ..., \sigma_n\}$ be the dual basis of $V^*$. Show that for any exterior $k$-form $\omega$ we have
\[ \sum_{i=1}^{n} \sigma_i \wedge (i_v, \omega) = k\omega. \]

Problem 3
(1) For a vector $v \in \mathbb{R}^3$ denote by $v^* := \langle v, \cdot \rangle \in (\mathbb{R}^3)^*$ where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. Show that for $v, w \in \mathbb{R}^3$ equipped with the standard scalar product
\[ *(v^* \wedge w^*) = (v \times w)^*. \]
(2) For a differentiable vector field $X$ on $\mathbb{R}^n$ denote by $X^*$ the corresponding differentiable 1-form with respect to the standard scalar product on all tangent spaces $T_p \mathbb{R}^n$.
(a) Compute $*d*(X^*)$. (Check if the expression is something you already know.)
(b) Let $n = 3$. Determine the vector field $Y$ which satisfies $*d(X^*) = Y^*$. (Have you seen the operation $X \mapsto Y$ before?)

The following problems will be discussed in the tutorials:

Problem 4
Show claim (3) from Satz 2.3. of the lecture that $\alpha \wedge * \beta = g(\alpha, \beta)dV$ for an $n$–dimensional oriented euclidean vector space $V$ with scalar product $g$, and $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^{n-k}(V^*)$.

Problem 5
Show statement (3) of Satz 2.5. of the lecture that $dF^*\alpha = F^*(d\alpha)$ for a diffeomorphism $F : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ and $\alpha \in \Omega^k(V)$. 