Klaus Mohnke
Institut für Mathematik
Rudower Chaussee 25
Haus 1 Raum 306

Homework Problems 7

Analysis and Geometry on Manifolds WS 06/07

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Problem 1
Show that the following definition is **equivalent** to Definition 1.14 of the lecture:
A manifold with boundary is a topological space M which admits a family of parametrizations $\{\phi_\alpha : U\alpha \to M\}$ such that for all α, β
(i) $U\alpha \subset \mathbb{H}^n$ and $\phi_\alpha(U\alpha) \subset M$ are open sets
(ii) $\cup_\alpha \phi_\alpha(U_\alpha) = M$
(iii) ϕ_α is a homeomorphism onto its image
(iv) $\phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(\phi_\alpha(U\alpha) \cap \phi_\beta(U_\beta)) \to \phi_\beta^{-1}(\phi_\alpha(U\alpha) \cap \phi_\beta(U_\beta))$ is differentiable.

Problem 2
(1) Show that the linear forms $\sigma^1, ..., \sigma^k$ on a vector space V are linear independent if and only if $\sigma^1 \wedge ... \wedge \sigma^k \neq 0$.
(2) Let $\{v_1, ..., v_n\}$ be a basis of V and $\{\sigma^1, ..., \sigma^n\}$ be the dual basis of V^*. Show that for for any exterior k-form ω we have

$$\sum_{i=1}^n \sigma^i \wedge (i_{v_i}\omega) = k\omega.$$

Problem 3
(1) For a vector $v \in \mathbb{R}^3$ denote by $v^* := \langle v, . \rangle \in (\mathbb{R}^3)^*$ where $\langle ., . \rangle$ denotes the standard scalar product. Show that for $v, w \in \mathbb{R}^3$ equipped with the standard scalar product

$$*(v^* \wedge w^*) = (v \times w)^*.$$

(2) For a differentiable vector field X on $\mathbb{R}^n$ denote by X^* the corresponding differentiable 1-form with respect to the standard scalar product on all tangent spaces $T_p\mathbb{R}^n$.
(a) Compute $*d*(X^*)$. (Check if the expression is something you already know.)
(b) Let $n = 3$. Determine the vector field Y which satisfies $*d(X^*) = Y^*$. (Have you seen the operation $X \mapsto Y$ before?)

The following problems will be discussed in the tutorials:

Problem 4
Show claim (3) from Satz 2.3. of the lecture that $\alpha \wedge *\beta = g(\alpha, \beta)dV$ for an n–dimensional oriented euclidean vector space V with scalar product g, and $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^{n-k}(V^*)$.

Problem 5
Show statement (3) of Satz 2.5. of the lecture that $dF^*\alpha = F^*(d\alpha)$ for a diffeomorphism $F : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ and $\alpha \in \Omega^k(V)$.