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Homework Problems 8

Analysis and Geometry on Manifolds WS 06/07

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Problem 1

(1) Let f be a differentiable function on \mathbb{R}^n . The *gradient* of f is the vector field $\nabla f \in \mathcal{X}(\mathbb{R}^n)$ which is uniquely defined by

$$\langle \nabla f_x, Y \rangle = d_x f(Y)$$

for $x \in \mathbb{R}^n$ and every $y \in T_x \mathbb{R}^n$. Let $c \in \mathbb{R}$ be a regular value of f : $d_x f : T_x \mathbb{R}^n \rightarrow \mathbb{R}$ is surjective for any $x \in f^{-1}(c) =: M_c$. Show that $\nabla f_x \perp T_x(M_c) \subset T_x \mathbb{R}^n$ for all $x \in M_c$.

(2) Show that there exists no immersion of a compact, differentiable manifold M without boundary into \mathbb{R}^n if $\dim M = n$.

(3) Show that there exists no injective differentiable map from \mathbb{R}^n to \mathbb{R}^k if $n > k$.

Problem 2

Symplectic Manifolds

(1) Let $\omega \in \Lambda^2(V)$ an exterior 2-form. Show that the following conditions are equivalent:

(i) ω is non-degenerate, i.e. the rank of its Gram matrix G is maximal: $2r = \text{rk} G = \dim V = n$.

(Notice that $\text{rk} G$ is always even. See Homework Set 6, Problem 4).

(ii) The induced linear map $v \in V \mapsto i_v \omega \in V^*$ is an isomorphism.

(iii) $\omega^r := \underbrace{\omega \wedge \cdots \wedge \omega}_r \neq 0 \in \Lambda^n(V)$.

(2) A *symplectic form* on a differentiable manifold is a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$: $d\omega = 0$, $\omega_x \neq 0$ for all $x \in M$. Let H be a differentiable function on M . Since ω is non-degenerate we can uniquely define a vector field $X_H \in \mathcal{X}(M)$ via

$$i_{X_H} \omega = -dH.$$

It is called the *symplectic gradient* of H . Assume that its flow, Φ_t exists globally on M (e.g. if M is compact). Show that $\Phi_t^* \omega = \omega$.

Problem 3

Let $\omega = \sum_i dp^i \wedge dq^i \in \Omega(\mathbb{R}^{2n})$ where $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$ with variable p^i and q^i for $i = 1, \dots, n$ on the first and second factor, respectively. Let $V = V(q) : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function and let

$$H = \frac{\|p\|^2}{2} + V(q)$$

(which is the sum of *kinetic and potential energy*). The gradient of V , $\nabla V(q)$ can be considered as the *conservative force* corresponding to the *potential* V . Compute the differential equation defining the flow-lines $\gamma(t) = (p(t), q(t))$ of X_H . Reformulate this equation purely in terms of a differential equation for $q = q(t)$.

The following problems will be discussed in the tutorials:

Problem 4

Let $\{U_i\}_{i \in I}$ be a countable, locally finite, open covering of a differentiable manifold M such that the closure $\overline{U_i}$ is compact for all i . Show that there exists an open covering $\{W_i\}_{i \in I}$ of M with the closures

$$\overline{W_i} \subset U_i.$$

Problem 5

(1) Show that for a differentiable family $\phi_t : M \rightarrow M$ of diffeomorphisms and a differentiable vector field Y we have

$$\frac{\partial}{\partial t}(d_{\phi_t(x)}\phi_t^{-1}(Y)) = d_{\phi_t(x)}\phi_t^{-1}(\mathcal{L}_{X_t}Y),$$

where $X_t(\phi_t(x)) := d\phi_t(x)/dt$.

(2) Show that for a differential form $\alpha \in \Omega^k(M)$ we have

$$\frac{\partial}{\partial t}(\phi_t^*\alpha) = \phi_t^*(\mathcal{L}_{X_t}\alpha)$$