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Homework Problems 9

Analysis and Geometry on Manifolds WS 06/07

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Merry Christmas and a Happy New Year!

Christmas problem
Let $c : [0, L] \to \mathbb{R}^2$ be a piecewise differentiable, simple closed curve, which parametrizes the boundary $\partial\Omega$ of a compact domain $\Omega \subset \mathbb{R}^2$ counterclockwise, and which is regular, i.e. $\dot{c}(t) \neq 0$ whereever it is defined (by assumption there are only finitely many times where it is not). Let $\theta := xdy \in \Omega^1(\mathbb{R}^2)$ where x, y are the standard coordinates.
(a) Compute $\int_0^L c^*\theta$ for the circle and a rectangle. Interpret the results geometrically and prove your reasonable conjecture for general domains Ω with the help of Stokes' Theorem.
(b) Use this conjecture to develop a formula for that geometric quantity for a polygon (in the plane) in terms of the coordinates of its vertices. Draw a beautiful Christmas star and apply this formula to it!
(c) Invent a machine which measures this quantity!

The following problems will be discussed in the tutorials:

Problem 2
(1) Show that a differentiable map $f : M \to N$ induces a homomorphism $f^* : H^k_{DR}(N) \to H^k_{DR}(M)$ via $f^[\alpha] := [f^*\alpha]$ for a closed differential form $\alpha \in \Omega^k(N)$ $(d\alpha = 0)$, where $[\alpha]$ denotes the equivalence class in $H^k_{DR}(M)$ represented by α and $f^*\alpha$ the pull-back. In particular, show that the map is well-defined.
(2) Let M, N be differentiable manifolds and $f, g : M \to N$ differentiable maps which are *homotopic*, i.e. there exists a differentiable map $F : [0,1] \times M \to N$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all $x \in M$. Construct linear maps $P : \Omega^k(N) \to \Omega^{k-1}(M)$ for any $k \geq 1$ such that

$$f^*\alpha - g^*\alpha = dP(\alpha) + P(d\alpha)$$

for all $\alpha \in \Omega^k(N)$ where we set $P \equiv 0$ on $\Omega^0(N)$. (Hint: Compare with the situation in the proof of Poincarés Lemma.)
(3) Use (2) to show that two maps, f and g, which are homotopic induce the same homomorphism on the cohomology:

$$f^* = g^* : H^k_{DR}(N) \to H^k_{DR}(M).$$

(4) Assume that the identity $\mathrm{id} : M \to M$ is homotopic to a constant map $M \to \{p\}$, $p \in M$.We say that M is contractible. Show that then $H^k_{DR}(M) = 0$ for all $k \geq 1$