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Homework Problems 10

Analysis and Geometry on Manifolds WS 06/07

due 18.1.2007

Problem 1 Let $F \subset \mathbb{R}^3$ be a differentiable surface with boundary, $(\mathbf{n}_F)_p$ the (oriented) normal vector to F for $p \in F$ and $(\mathbf{n}_{\partial F})_p$ be the outward normal at $p \in \partial F$. For a vector field X on $\mathbb{R}^3$ denote by $rot(X)$ its rotation. (see Lecture for precise definitions). Prove the classical Stokes-theorems:
(a)

$$\int_F \langle rot(X), \mathbf{n}_F \rangle dF = \int_{\partial F} \langle X, \mathbf{d}(\partial F) \rangle$$

where the one-form over which is integrated on the right hand side is given by $\gamma^* \langle X, \mathbf{d}(\partial F)\rangle := \langle X(\gamma(t)), \dot{\gamma}_i(t)\rangle dt$ for any differentiable map $\gamma : (a, b) \to \partial F$.
(b)

$$\int_F div_F(X) dF = \int_{\partial F} \langle F, \mathbf{n}_{\partial F} \rangle d(\partial F)$$

where $div_F(X)$ is (uniquely) defined by $div_F(X)dF = d(i_X dF)$ (analogous to the divergence on $\mathbb{R}^3$).

Problem 2 Let $W \subset \mathbb{R}^3$ be an open subset, $\phi : W \to \mathbb{R}$ be a differentiable function and V be a differentiable vector field on W. Let $p \in W$ be a point. Show by using Stokes' theorem that

$$divV(p) = \lim_{\epsilon \to 0} \frac{3}{4\pi\epsilon^3} \int_{\partial B_\epsilon(p)} \beta_V$$

where $\beta_V := i_V dx^1 \wedge dx^2 \wedge dx^3$ and

$$X_p(\phi) = \lim_{\epsilon \to 0} \frac{3}{4\pi\epsilon^3} \int_{\partial B_\epsilon(p)} \phi \beta_X$$

for any constant vector field X on W.

Problem 3 (a) Let (M, g) be an oriented Riemannian manifold with boundary, dM and $d(\partial M)$, the associated volume forms of M and its boundary ∂M, i.e. $d(\partial M) = i_{\mathbf{n}} dM$, where $\mathbf{n}$ is the outward normal along the boundary. For a differentiable vector field V on M define its *divergence*, $divV$ by

$$(divV)dM := d(i_V dM).$$

Prove the general divergence theorem

$$\int_M divV dM = \int_{\partial M} g(V, \mathbf{n}) d(\partial M).$$

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(b) Define, moreover, the Laplacian of a differentiable function u via

$$\Delta u = div(\nabla u)$$

where ∇u is the *gradient* of u defined by $g(\nabla u, X) = du(X)$. Prove the general Green formulas

$$\begin{aligned}\int_M \Delta u dM &= \int_{\partial M} g(\nabla u, \mathbf{n}) d(\partial M) \\ \int_M (u\Delta v - \Delta u v) dM &= \int_{\partial M} (ug(\nabla v, \mathbf{n}) - g(\nabla u, \mathbf{n})v) d(\partial M).\end{aligned}$$

The following problem will be discussed in the tutorials:

Problem 4
(1) (See Homework Set 8, Problem 2 for the definitions.) Let $\omega \in \Omega^2(M)$ be a symplectic structure on the even-dimensional differentiable manifold M. Let us define the following binary operation on $C^\infty(M)$: For differentiable functions $f, g \in C^\infty(M)$, their *Poisson bracket* $\{f, g\} \in C^\infty(M)$ is given by

$$\{f, g\} := \omega(X_f, X_g).$$

Prove the following identities:
(i) $\{f, g\} = -\{g, f\}$
(ii) $\{f, gh\} = \{f, g\}h + g\{f, h\}$.
(iii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.
Such a structure is called a *Poisson algebra*.
(2) Show that the map $f \in C^\infty(M) \mapsto X_f \in \mathcal{X}(M)$ satisfies $[X_f, X_g] = X_{\{f,g\}}$.