Klaus Mohnke
Institut für Mathematik
Rudower Chaussee 25
Haus 1 Raum 306

Homework Problems 11

Analysis and Geometry on Manifolds WS 06/07

due 25.1.2007

Problem 1
(a) Show that the cone in $\mathbb{R}^3$ formed by the rays starting at $(0,0,c)$ for $c \geq 0$ and going through a point on the unit circle in $\mathbb{R}^2 \times 0$ (with its cone point deleted) is locally isometric to $\mathbb{R}^2$, i.e. for each point there is a neighbourhood which admits coordinates (x_1, x_2) such that

$$g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}$$

where g is the Riemannian structure induced by the standard euclidean product on $\mathbb{R}^3$. *Note: A heuristic argument is considered sufficient and helpful for (b).*
(b) Determine c such that for all pairs (p,q) the minimum of the length functional is achieved, i.e. there is a geodesic γ from p to q such that $\ell(\gamma) = d(p,q)$. For which parameters c are there locally minimizing non-constant geodesics connecting a point p with itself?
$(c)^*$ Determine c such that there are two, three,...,n different locally minimizing non-constant geodesics connecting p to itself.

Problem 2
Let $M \subset \mathbb{R}^N$ be a differentiable submanifold, its Riemannian metric given by the standard euclidean structure on $\mathbb{R}^N$.
(a) Show that normalized geodesics $\gamma = \gamma(t)$ on (M,g) are characterized by $\ddot{\gamma}(t) \perp T_{\gamma(t)}M \subset \mathbb{R}^N$. Give also a heuristic explanation for this fact.
(b) Determine the geodesics of the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

Problem 3
Consider $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the Riemannian metric

$$g = \frac{1}{y^2} g_0$$

where g_0 is the standard euclidean structure on $\mathbb{R}^2$. Show that the set of geodesics of $(\mathbb{H}, g)$ is given by half circles with centers on $\partial\mathbb{H} = \{(x,0) \mid x \in \mathbb{R}\}$ and half lines perpendicular to $\partial\mathbb{H}$. *Hint: The concrete parametrization does not matter in this context. Construct the circle (or the line) out of the initial data of the second order differential equation (see Problem 5) and show that the geodesic stays on it).*

Bitte wenden...

The following problems will be discussed in the tutorials:

Problem 4

Show that the differential equation defining *geodesics* on a Riemannian manifold is independent of the chosen coordinates. In particular, show that the coordinates with respect to one parametrization satisfy the corresponding equations if and only if the coordinates with respect to another parameterization satisfy the equations corresponding to these coordinates.

Problem 5

Justify the normalization of the minimal, regular curve $\gamma : [a, b] \to M$, i.e. γ differentiable and $\dot{\gamma} \neq 0$ everywhere, such that $\|\dot{\gamma}\|_g \equiv 1$ where $\|.\|_g$ is the norm of the tangent vector with respect to the Riemannian metric g which we did in the derivation of the equation for geodesics in the lecture. That means: show that for a solution to the differential equation

$$\ddot{\gamma}^k + \sum_{l,i,j} \frac{1}{2} g^{kl} \Big(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \Big) \dot{\gamma}^i \dot{\gamma}^j$$

the norm $\|\dot{\gamma}(t)\|_g$ is constant in t. $g_{kl} = g_{kl}(\gamma(t))$ are the entries of the Gram matrix of g with respect to a set of chosen coordinates, g^{kl} the entries of its inverse, $\dot{\gamma}^k, \ddot{\gamma}^k$ the coordinates of $\dot{\gamma}, \ddot{\gamma}$.