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Homework Problems 13

Analysis and Geometry on Manifolds WS 06/07

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Problem 1 *Covariant derivative along a map*
Let ∇ be a covariant derivative acting on vector fields on a manifold M.
(1) For a 1-form $\alpha \in \Omega^1(M)$ and a tangent vector $X \in T_pM$ define $\nabla_X \alpha \in T_p^*M$ via

$$X(\alpha(Y)) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y)$$

for any differentiable vector field Y (see Set 12, Problem 2). Show that $(X, \alpha) \mapsto \nabla_X \alpha$ is a bilinear map which satisfies Leibniz' rule

$$\nabla_X(f\alpha) = X(f)\alpha + f\nabla_X \alpha$$

for any differentiable function f on M.
(2) Let $u : F \to M$ be a differentiable map between manifolds. A *vector field along* u is a differentiable family $\{Y(z)\}_{z\in F}$ of tangent vectors $Y(z) \in T_{u(z)}M$. Differentiability refers to differentiable coefficients in the decomposition $Y(z) = \sum_i Y^i \frac{\partial}{\partial x_i}$ with respect to any set of coordinates on M. Notice that Y^i are differentiable functions on (open subsets of) F. The covariant derivative of a vector field Y along u in direction of a tangent vector $v \in T_zF$, $\nabla^u_v Y$, should satisfy

$$v(\alpha(Y)) = (\nabla_{d_z u(v)} \alpha)(Y) + \alpha_{u(z)}(\nabla^u_v Y)$$

for any 1-form $\alpha \in \Omega^1(M)$, where $\alpha(Y)(z) := \alpha_{u(z)}(Y(z))$.
(a) Show that this requirement determines $\nabla^u_v Y$ completely.
(b) Show that $(v, Y) \mapsto \nabla^u_V Y$ is a bilinear map.
(c) Show that ∇^u satisfies Leibniz' rule: $\nabla^u_v(fY) = v(f)Y + f\nabla^u Y$ for any differentiable function f
(d) Show that ∇^u is metric provided that ∇ is: $v(g(Y,Z)) = g(\nabla^u_v Y, Z) + g(Y, \nabla^u_v Z)$.
(e) Show that ∇^u is *torsion free* if ∇ is: $\nabla^u_{d_z u(X)} Y - \nabla^u_{d_z u(Y)} X - d_z u([X,Y]) = 0$ for all differentiable vector fields X, Y on F
(f) Express ∇^u in local coordinates, i.e. compute

$$\nabla_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial x_i} =: \sum_k a^k_{\alpha i} \frac{\partial}{\partial x_k}$$

for coordinates z_α on F and x_i on M. Explain why this determines ∇^u and why its representation via $a^k_{\alpha i}$ does not depend on the coordinates chosen.

Problem 2
Determine the parallel transport on $S^2 \subset \mathbb{R}^3$ along the closed paths given by:
(a) three quarter segments of grand circles forming a triangle
(b) an arbitrary circle.

Bitte wenden...

The following problems will be discussed in the tutorials:

Problem 3
Let M be a smooth manifold. Define the canonical 1-form θ on T^*M via

$$\theta_\alpha(X) := \alpha(d\pi_\alpha(X))$$

where $\alpha \in T^*M$, $\pi : T^*M \to M$ is the projection assigning the base point to the cotangent vector, $X \in T_\alpha(T^*M)$. Show that $d\theta \in \Omega^2(T^*M)$ is a symplectic form (Hint: Use local coordinates).

Problem 4
Let (M, g) be a Riemannian manifold. g induces a linear identification $T_pM \cong T_p^*M$ via $X \mapsto g(X, .)$. Hence we can measure lengths of elements in T_p^*M with the help of g. Let $H : T^*M \to \mathbb{R}$ be given by $H(\alpha) = \|\alpha\|^2/2$. Determine the equation for the Hamiltonian flow of H. Reformulate this equation purely as an equation for curves in M (rather than T^*M).