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Homework Problems 14

Analysis and Geometry on Manifolds WS 06/07

Food for Thought during Semester Break

Problem 1 *Curvature tensor and Bianchi identities*
Let ∇ be a covariant derivative acting on vector fields of a manifold. Consider the map $(X, Y, Z) \mapsto R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ where X, Y, Z are differentiable vector fields on M. Notice that the range of the map are vector fields on M.
(1) Show that this map is linear and *tensorial*, the latter meaning

$$fR(X, Y)Z = R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ)$$

for any differentiable function f on M. In particular, $(R(X, Y)Z)(p)$ depends only on the values of X, Y, Z at p.
(2) Show that $R(X, Y)Z = -R(Y, X)Z$.
(3) (1st Bianchi identity) If ∇ is torsion free show that $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.
(4) If ∇ is metric with respect to a Riemannian metric g on M show that $g(R(X, Y)Z, U) + g(Z, R(X, Y)U) = 0$ for another vector field U.
(5) (2nd Bianchi identity) Since R is tensorial we may interpret $R(X, Y)$ as family of endomorphisms of TM: $R(X, Y)(p) \in End(T_pM)$. Define $\nabla_Z R(X, Y) \in End(TM)$ via $\nabla_Z R(X, Y)U := \nabla_Z(R(X, Y)U) - R(X, Y)(\nabla_Z U)$. Show that (without any assumption) $\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) - R([X, Y], Z) - R([Y, Z], X) - R([Z, X], Y) = 0$
(6) Define R^i_{jkl} via

$$R(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k})\frac{\partial}{\partial x_l} = \sum_{i=1}^{n} R^i_{jkl} \frac{\partial}{\partial x_i}$$

for a chosen set of coordinates $\{x_i\}$. Express all the identities above with respect to these coordinates.

Problem 2 *Tubular neighbourhood theorem*
Let $M \subset N$ be a closed differentiable submanifold. Fix a Riemannian metric g on N. For $p \in M$ let $\nu(M)_p := \{v \in T_pN \mid g(v, T_pM) = 0\}$ denote the *normal bundle* of M in N. Show that there is a neighborhood $N(M) \subset N$ and a diffeomorphism $\Phi : \nu(M) \to N(M)$ such that $\Phi(0_p) = p$ for the zero tangent vector $0_p \in T_p$.
Hint: First convince yourself that for any $\epsilon > 0$ $\nu(M)$ is diffeomorphic to $\{v \in \nu(M) \mid \|v\| < \epsilon\}$. Then use the exponential map, the inverse function theorem,...
If you have time left, try to prove the statement for any non-compact M without boundary.