Problem 1  Curvature tensor and Bianchi identities
Let $\nabla$ be a covariant derivative acting on vector fields of a manifold. Consider the map $(X, Y, Z) \mapsto R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ where $X, Y, Z$ are differentiable vector fields on $M$. Notice that the range of the map are vector fields on $M$.

(1) Show that this map is linear and tensorial, the latter meaning 
$$fR(X, Y)Z = R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ)$$
for any differentiable function $f$ on $M$. In particular, $(R(X, Y)Z)(p)$ depends only on the values of $X, Y, Z$ at $p$.

(2) Show that $R(X, Y)Z = -R(Y, X)Z$.

(3) (1st Bianchi identity) If $\nabla$ is torsion free show that 
$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$ 

(4) If $\nabla$ is metric with respect to a Riemannian metric $g$ on $M$ show that $g(R(X, Y)Z, U) + g(Z, R(X, Y)U) = 0$ for another vector field $U$.

(5) (2nd Bianchi identity) Since $R$ is tensorial we may interpret $R(X, Y)$ as family of endomorphisms of $TM$: $R(X, Y)(p) \in \text{End}(T_p M)$. Define $\nabla_Z R(X, Y) \in \text{End}(TM)$ via $\nabla_Z R(X, Y)U := \nabla_Z (R(X, Y)U) - R(X, Y)(\nabla_Z U)$. Show that (without any assumption) 
$$\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) - R([X, Y], Z) - R([Y, Z], X) - R([Z, X], Y) = 0.$$ 

(6) Define $R^j_{\ell kl}$ via 
$$R(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) \frac{\partial}{\partial x_\ell} = \sum_{i=1}^n R^i_{jkl} \frac{\partial}{\partial x_i}$$
for a chosen set of coordinates $\{x_i\}$. Express all the identities above with respect to these coordinates.

Problem 2  Tubular neighbourhood theorem
Let $M \subset N$ be a closed differentiable submanifold. Fix a Riemannian metric $g$ on $N$. For $p \in M$ let $\nu(M)_p := \{ v \in T_p N \mid g(v, T_p M) = 0 \}$ denote the normal bundle of $M$ in $N$. Show that there is a neighborhood $N(M) \subset N$ and a diffeomorphism $\Phi: \nu(M) \to N(M)$ such that $\Phi(0_p) = p$ for the zero tangent vector $0_p \in T_p$.

Hint: First convince yourself that for any $\epsilon > 0$ $\nu(M)$ is diffeomorphic to $\{ v \in \nu(M) \mid \|v\| < \epsilon \}$. Then use the exponential map, the inverse function theorem,...

If you have time left, try to prove the statement for any non-compact $M$ without boundary.