Vector fields, flows

November 23, 2006

Let X, Y be smooth vector fields on a manifold M, with flows ϕ_t and ψ_s, supposed complete. Show that the following statements are equivalent :

1. $[X, Y] = 0$
2. $X(f \circ \psi_s) = X(f) \circ \psi_s$ for any $f \in C^\infty(M)$ and all $s \in \mathbb{R}$.
3. $[\phi_t, \psi_s] := \phi_t \circ \psi_s \circ \phi_t^{-1} \circ \psi_s^{-1} = \mathbb{I}_M$ for all $t, s \in \mathbb{R}$.

Proof. We first prove $(1) \Leftrightarrow (2)$ and then $(2) \Leftrightarrow (3)$

- The statement 2 is equivalent to $X(f \circ \psi_s) \circ \psi_s^{-1}(m) = X(f)$ for all f, m, s. Denote by G the function

  $$G(s) = X(f \circ \psi_s) \circ \psi_s^{-1}(m) - X(f)(m).$$

  Statement (2) is equivalent to $G(s) \equiv 0$. One has obviously $G(0) = 0$. On the other hand, one has the derivative

  $$\left.\frac{dG}{ds}\right|_s = X(Y(f)\psi_s) \circ \psi_s^{-1}(m) - Y(X(f \circ \psi_s)) \circ \psi_s^{-1}(m).$$

  Using $Y(f) \circ \psi = Y(f \circ \psi)$ one obtains

  $$\left.\frac{dG}{ds}\right|_s = [X, Y](f \circ \psi_s) \circ \psi_s^{-1}(m).$$

  Suppose (2). This implies in particular that $\left.\frac{dG}{ds}\right|_s = 0$ for all s, and therefore (1). Conversely, if $[X, Y] = 0$, one has $\left.\frac{dG}{ds}\right|_s = 0$ for all s, and this ODE with the "initial" condition $G(0) = 0$ is solved uniquely by $G(s) = 0$ for all s, i.e. statement (2).

- Now, the statement (3) is equivalent to

  $$F(t, s) := f \circ \phi_t \circ \psi_s \circ \phi_t^{-1}(m) - f \circ \psi_s(m) = 0 \, \forall t, s$$

  for all $f \in C^\infty(M)$ and all $m \in M$. On the other hand, one has always the derivatives

  $$\frac{\partial F}{\partial t} = X(f) \circ \phi_t \circ \psi_s \circ \phi_t^{-1}(m) - X(f \circ \phi_t \circ \psi_s) \circ \phi_t^{-1}(m).$$

Now, if we use $X(f) \circ \phi_t = X(f \circ \phi_t)$ and assume (2), i.e. $X(f \circ \phi_t) \circ \psi_s = X(f \circ \phi_t \circ \psi_s)$, we obtain

$$\frac{\partial F}{\partial t} = X(f \circ \phi_t \circ \psi_s) \circ \phi_t^{-1}(m) - X(f \circ \phi_t \circ \psi_s) \circ \phi_t^{-1}(m) = 0$$

for all t, s. Moreover, for $t = 0$ and any s, one has $F(0, s) = 0$. With s fixed, this is an ODE whose solution is $F(t, s) = 0$ for all t, which implies (3). Conversely, assuming (3), one has $\frac{\partial F}{\partial t} = 0$ for all t, and in particular for $t = 0$, which provides precisely $X(f) \circ \psi_s(m) - X(f \circ \psi_s)(m) = 0$, i.e. the statement (2).

$\square$