# Übungen zur Analysis für Physiker

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**Übungsblatt 4 bis 5**

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### Exercise 1: Time functions?

Which of the following functions are time functions on the respective domain of definition? Which are temporal? Cauchy? Steep?

1. $x_0|_U$ for $U := J^+((-1,0)) \cap J^-((1,0)) \subset \mathbb{R}^{1,n}$;
2. $x_0^3$ on $\mathbb{R}^{1,n}$;
3. $\arctan \circ x_0$ on $\mathbb{R}^{1,n}$;
4. $x_0 + \frac{1}{10} \cdot (\arctan \circ x_0) \cdot x_1$ on $(\mathbb{R}^2, e^{\arctan \circ (x_0+x_1)} g_{1,1})$.

### Exercise 2: The symbol

Let a partial differential operator $A$ of order $\ell$ between vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$ be given. We want to define the principal symbol of $A$ as a totally symmetric vector bundle homomorphism from $\bigotimes_{i=1}^k \tau^* M$ to $\pi_E^* \otimes \pi_F$. Given a frame $\partial_1, ..., \partial_n$ in $x \in M$, we use the notation $\xi_i := \partial_1^{i_1} \otimes ... \otimes \partial_n^{i_k} \in \bigotimes_{k=1}^{|i|} T_x M$ for a multiindex $i = (i_1, ..., i_k)$. Show that the following two characterizations of the term 'principal symbol' are well-defined and equivalent, possibly up to overall constants:

1. If $A$ is written w.r.t. trivializing local coordinate charts $\kappa_E$ of $\pi_E$, $\kappa_F$ of $\pi_F$ as

   $$\kappa_F^{-1} \circ A \circ (\cdot \circ \kappa_E) = \sum_{i \text{ multiindex}, |i| \leq \ell} A_i \partial_i$$

   for matrices $A_i$, then the principal symbol $\sigma(A)$ of $A$ is defined by

   $$\kappa_F^{-1} \circ \sigma(A) \circ (\cdot \circ \kappa_E) = \sum_{i \text{ multiindex}, |i| = \ell} A_i \xi_i.$$

2. Let $f \in C^\infty(M, \mathbb{R})$ vanish at $x \in M$. The principal symbol of $A$ at $x \in M$ is the unique multilinear bundle map $\sigma_x(A)$ from $T_x^* M$ to $\pi_1^* \otimes \pi_2$ such that $\sigma(A)(d_x f, ..., d_x f)(\psi(x)) = (A(f^\ell \cdot \psi))(x)$ for any local section $\psi$ around $x$.

### Exercise 3: Calculating symbols

1. Show: $[\nabla_X, A]$ is of order $\leq \ell$ for any linear differential operator $A$ of order $\leq \ell$.
2. Show $\sigma(A \circ B) = \sigma(A) \circ \sigma(B)$ for any two partial differential operators $A, B$.
3. Show that for $d$ being the exterior derivative we get $\sigma(d)(x)(v) = v^b \wedge \cdot$.
4. Show that $\sigma(\mathrm{tr}^g(\nabla^2)) = g \otimes \mathbf{1}$.