Übungen zur Lorentzgeometrie und Mathematischen Relativitätstheorie

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Exercise 1: Killing vector fields along timelike curves

Let X be a timelike Killing vector field on a spacetime (M,g), and let c be a timelike curve parametrized by Lorentzian arclength on an interval $I=[a,b)$. We consider the real functions E and a on I defined by $E := -g(c',X)$ and $a := \sqrt{g(c'',c'')}$.

1. Show that a is well-defined (sign under the square-root!) and that $|E'| \leq a \cdot E$.
2. Show that if $\int_I a < \infty$ then $\lim_{t \to b} E(t) < \infty$.
3. Show that if $\int_I a < \infty$ then $t \mapsto |g(X(c(t),X(c(t)))|$ is bounded on I.

Remark. The relevance of the last item is the following: In some spacetimes, like in the so-called negative-mass Schwarzschild spacetimes, which carry a timelike Killing vector field X, there are timelike curves of finite length along which $g(X,X)$ tends to $-\infty$. Item 3 says that those curves cannot be trajectories of autonomous rockets, as those have positive mass without fuel and can carry only a finite amount of fuel to generate acceleration.

Exercise 2: Spatial compactness

Let (M,g) be globally hyperbolic and let $S \subset M$ be a smooth Cauchy surface of (M,g). Let $K \subset M$ and $C \subset S$ be compact.

1. Show that $J^+(K) \cap J^-(S)$ is compact.
2. Let P be a symmetric-hyperbolic operator on $\pi : E \to M$. Show that, for any solution $u \in \Gamma_{C^\infty}(\pi)$ of $Pu = 0$ with $\mathrm{supp}(u|_S) \subset C$, its support $\mathrm{supp}(u)$ is spatially compact, but not compact.

Exercise 3: Maxwell theory

Let (M,g) be a semi-Riemannian manifold of signature (r,s). For $\alpha \in \Omega^k(M)$ we define $*\alpha \in \Omega^{n-k}$ by $\beta \wedge *\alpha = g(\beta,\alpha) \cdot \mathrm{vol}$ for every $\beta \in \Omega^k(M)$, where g is the extension of the metric to Ω^k (as in the DG Primer). We get $*^2 = (-1)^{k(n-k)+r}\mathbf{1}$. The **formal adjoint** of an operator $A : \Gamma(\pi) \to \Gamma(\pi)$ is defined via the equality $\int_M \langle a, A^* b\rangle = \int_M \langle Aa, b\rangle$ for any two compactly supported smooth $a, b \in \Gamma(\pi)$. We will see in the lecture that $d_k^* = (-1)^{nk+1+r} * d * : \Omega^{k+1}(M) \to \Omega^k(M)$. We define $P : \Omega^1(M) \to \Omega^1(M)$ by $P(\alpha) := d^* d\alpha = (-1)^{n+r+1} * d *$ for all $\alpha \in \Omega^1(M)$. This operator is called **Maxwell operator**.

1. Show that Lorentzian $(r=1)$ g.h. case, P has no well-posed initial value problem.
2. Let, in the four-dimensional Lorentzian case $r=1, s=3$, (dx_0, dx_1, dx_2, dx_3) be an oriented coordinate base g-pseudo-orthonormal at x. Calculate $*(dx_i \wedge dx_j)$ for $i,j \in \mathbb{N}_3$ at x.
3. Show that in the Lorentzian $(r=1)$ g.h. case, for every $\alpha \in \Omega^1(M)$ there is $f \in C^\infty(M)$ with $d^*\tilde{\alpha} = 0$ for $\tilde{\alpha} := \alpha + df$. Show that $P\alpha = P\tilde{\alpha} = \tilde{P}\tilde{\alpha}$ for $\tilde{P} := d^*d + dd^*$. Show that $\tilde{P}$ is symmetric hyperbolic. You might restrict to four dimensions.