Seminar on Lorentzian Geometry and Mathematical General Relativity

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General Relativity (GR) and Quantum Field Theory (QFT), more explicitly, the Standard Model of particle physics, are the two pillars of fundamental physics, and until today reconciling them in a unified model remains an unresolved task. Whereas QFT alone struggles with severe conceptual gaps (e.g., a clear understanding of the measurement process) and mathematical inconsistencies (non-perturbative formulation for nonlinear field theories, quantum fields in in curved spacetimes, convergence of perturbative expansions), GR is a closed theory that nevertheless predicts paradoxical results like black holes: Regions of spacetime from which every possible movement ends in finite time at the timelike boundary of the universe.

This seminar shall serve as a first glance on current topics of research in GR and its underlying mathematical disciplines, Lorentzian geometry and hyperbolic PDEs.

Required previous knowledge comprises Differential Geometry 1 and, ideally, Partial Differential Equations 1.

The seminar is divided in two major parts: The first considers some topics relevant for moduli spaces of GR solutions: energy conditions, initial value constructions, submanifolds of prescribed mean curvature, conformal boundaries, and possibly the positive mass theorem. The second considers the reformulation of GR in a synthetic context.

The majority of talks is independent of each other, and independent of the course. Intersections with topics of the course occur in Talks 1b, 1c, 1d. The script of the course can be used as a general background reference, along with [2], [5] and [19]. Optional talks are marked by an asterisk (*).

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Seminar plan with recommended literature:

1. **Introductory talk: Lorentzian Geometry. A world of wonder** (Olaf Müller)

2. **Part 1: Hot topics in classical mathematical GR**

   (a) **Talk 1(*):** *Bounds on sectional curvature and Comparison Theory in Lorentzian signature.* This talk presents results about Lorentzian comparison theory as in [5] and [19] and the result that two-sided bounds on sec imply constancy of sec in Lorentzian signature ([13]).

   (b) **Talk 2:** Einstein equations, energy conditions, splitting theorems [11]

   (c) **Talk 3(*):** The Kerr-Newman family and its geodesics [19], [20]

   (d) **Talk 4:** Hyperbolic equations in globally hyperbolic spacetimes, symplectic and Kähler structures on spaces of solutions [2], [3], [22], [15], [16]

   (e) **Talk 5(*):** The Lorentzian index theorem [4]

   (f) **Talk 6:** Techniques for construction of initial values. Here the *conformal method* ([7]) and the *gluing method* ([9]) are to be presented

   (g) **Talk 7:** Submanifolds of prescribed mean curvature in Lorentzian signature, mainly based on [12]

   (h) **Talk 8:** Lorentzian conformal geometry, invariance of null pregeodesics, and the conformal boundary [10], [1]

3. **Part 2: Synthetic Lorentzian geometry**

   (a) **Talk 9:** Abstract boundaries, used in a black-hole theorem [17], [8]

   (b) **Talk 10(*):** *Lorentzian-Riemannian functor and finiteness theorem* [18]

   (c) **Talk 11:** Riemannian interlude: Optimal transport and Ricci bounds in metric measure spaces [21]

   (d) **Talk 12:** Synthetic Lorentzian geometry, the Einstein equation as optimal transport, and synthetic singularity theorems ([6])

4. **Wrap-up and outlook** (Olaf Müller)

References

[1] Michael Anderson, Piotr Chrusciel: *Asymptotically simple solutions of the vacuum Einstein equations in even dimensions.* Communications in Mathematical Physics 260(3):557-577. arXiv:gr-qc/0412020

[2] Christian Bär: Lorentzgeometrie. Vorlesungsskript, `https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/skript-LorGeo.pdf`

[3] Christian Bär: *Green-hyperbolic operators on globally hyperbolic spacetimes*, Commun. Math. Phys. 333, 1585-1615 (2015). arXiv:1310.0738

[4] Christian Bär, Alexander Strohmaier: *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*, American Journal of Mathematics 141 (5), 1421 — 1455 (2019). arXiv:1506.00959

[5] J. Beem, P.Ehrlich, K.Easley: *Global Lorentzian geometry.* DOI:10.1201/9780203753125

[6] Fabio Cavalletti, Andrea Mondino: *A review of Lorentzian synthetic theory of timelike Ricci curvature bounds.*arXiv:2204.1333

[7] Yvonne Choquet-Bruhat, James Isenberg, and James W. York, Jr.: *Einstein constraints on asymptotically Euclidean manifolds*, Phys. Rev. D 61, 084034 (2000)

[8] Ivan Costa e Silva, José Luis Flores, Jonatán Herrera: *Hausdorff closed limits and the c-boundary II: null infinity and black holes*, Class. Quantum Grav. 36 185007. arXiv:1807.00152

[9] Corvino J, Schoen RM: *On the asymptotics for the vacuum Einstein constraint equations.* J Differ Geom 73:185 —217 (2006)

[10] Helmut Friedrich: *Cauchy problems for the conformal vacuum field equations in general relativity*, Commun. Math. Phys. 91 (1983), 445 — 472.

[11] Greg Galloway: *Some rigidity results for spatially closed spacetimes*, Mathematics of gravitation, Part I (Warsaw, 1996), Banach Center Publ., vol. 41, Polish Acad. Sci., Warsaw, 1997, pp. 21 — 34 `http://matwbn.icm.edu.pl/ksiazki/bcp/bcp41/bcp4112.pdf`

[12] Claus Gerhardt: *Hypersurfaces of prescribed mean curvature in Lorentzian manifolds.* Mathematische Zeitschrift 235, pp 83 — 97 (2000)

[13] Steven Harris: *A triangle comparison theorem for Lorentz manifolds.* Indiana Univ. Math. J. 31, no. 3, 289308 (1982)

[14] Michael Kunzinger, Clemens Sämann: *Lorentzian length spaces.* Ann. Global Anal. Geom. 54, no. 3, 399 — 447 (2018). arXiv:1711.08990

[15] Albert Much, Robert Oeckl: *Self-Adjointness in Klein-Gordon Theory on Globally Hyperbolic Spacetimes*. arXiv:1804.07782

[16] Albert Much, Robert Oeckl: *Complex Structures for Klein-Gordon Theory on Globally Hyperbolic Spacetimes*. arXiv:1812.00926

[17] Olaf Müller: *Topologies on the future causal completion*. arXiv:1909.03797

[18] Olaf Müller: *Lorentzian Gromov-Hausdorff theory and finiteness results*. arXiv:1912.00988

[19] Barrett O'Neill: *Semi-Riemannian geometry: With applications to relativity*, Pure and Applied Mathematics, vol. 103, Academic Press, ISBN 0-12-526740-1 (1983)

[20] Barrett O'Neill: *The Geometry of Kerr Black Holes*, A K Peters, Ltd, 289 Linden Street, Wellesley, MA 02181, ISBN: 1-56881-019-9 (1995)

[21] Cédric Villani: Optimal transport — Old and new. Springer-Verlag (2008)

[22] N.M.J. Woodhouse: Geometric quantization. Oxford Mathematical Monographs. Clarendon Press (1997)