Seminar on Lorentzian Geometry and Mathematical General Relativity

Olaf Müller*

April 15, 2024

General Relativity (GR) and Quantum Field Theory (QFT), more explicitly, the Standard Model of particle physics, are the two pillars of fundamental physics, and until today reconciling them in a unified model remains an unresolved task. Whereas QFT alone struggles with severe conceptual gaps (e.g., a clear understanding of the measurement process) and mathematical inconsistencies (non-perturbative formulation for nonlinear field theories, quantum fields in in curved spacetimes, convergence of perturbative expansions), GR is a closed theory that nevertheless predicts paradoxical results like black holes: Regions of spacetime from which every possible movement ends in finite time at the timelike boundary of the universe.

This seminar shall serve as a first glance on current topics of research in GR and its underlying mathematical disciplines, Lorentzian geometry and hyperbolic PDEs.

Required previous knowledge comprises Differential Geometry 1 and, ideally, Partial Differential Equations 1.

The seminar is divided in three major parts: The first two considers some geometric resp. analytic topics relevant for moduli spaces of GR solutions: energy conditions, initial value constructions, submanifolds of prescribed mean curvature, conformal boundaries. The third considers the reformulation of Lorentzian geometry (and ultimately of GR) in a synthetic manner.

The majority of talks is independent of each other, and independent of the course. Intersections with topics of the course occur in Talk 2. The script of the course can be used as a general background reference, along with [2], [5] and [19]. Optional talks are marked by an asterisk (*).

*Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin (Germany), Email: o.mueller@hu-berlin.de

Seminar plan with recommended literature:

1. **Introductory talk: Lorentzian Geometry. A world of wonder** (Olaf Müller)
2. **Part 1: Geometric aspects of general relativity**
   (a) **Talk 1:** *Bounds on sectional curvature and Comparison Theory in Lorentzian signature.* This talk presents results about Lorentzian comparison theory as in [5] and [19] and the result that two-sided bounds on sec imply constancy of sec in Lorentzian signature ([15]).
   (b) **Talk 2:** Einstein equations, energy conditions, splitting theorems [12]
   (c) **Talk 3:** The Kerr-Newman family and its geodesics [19], [20]
3. **Part 2: Analytic aspects of general relativity**
   (a) **Talk 5(*):** The Lorentzian index theorem [3]
   (b) **Talk 6:** Techniques for construction of initial values. Here the *conformal method* ([7]) and the *gluing method* ([10]) are to be presented
   (c) **Talk 7:** Submanifolds of prescribed mean curvature, in particular maximal hypersurfaces, in Lorentzian signature, mainly based on [13]
   (d) **Talk 8:** Lorentzian conformal geometry, the Malament-Hawking-King-McCarthy Theorem, invariance of null pregeodesics, and the conformal boundary [11], [1]
   (e) **Talk 9:** The appearance of trapped surfaces and black holes caused by concentration of energy [21]
   (f) **Talk 10:** Global existence of Yang-Mills solutions [8]
4. **Part 3: Synthetic Lorentzian geometry**
   (a) **Talk 11:** Abstract boundaries, used in a black-hole theorem [17], [9]
   (b) **Talk 12:** Lorentzian (length) spaces [16], [4]
   (c) **Talk 13:** Causal fermion systems as a model of quantum gravity [6]
5. **Wrap-up and outlook** (Olaf Müller)

References

[1] Michael Anderson, Piotr Chrusciel: *Asymptotically simple solutions of the vacuum Einstein equations in even dimensions.* Communications in Mathematical Physics 260(3):557-577. arXiv:gr-qc/0412020

[2] Christian Bär: Lorentzgeometrie. Vorlesungsskript, `https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/skript-LorGeo.pdf`

[3] Christian Bär, Alexander Strohmaier: *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*, American Journal of Mathematics 141 (5), 1421 — 1455 (2019). arXiv:1506.00959

[4] Annegret Burtscher, Leonardo Garcia-Heveling: *Time functions on Lorentzian length spaces.* arXiv:2108.02693

[5] J. Beem, P.Ehrlich, K.Easley: *Global Lorentzian geometry.* DOI:10.1201/9780203753125

[6] `https://causal-fermion-system.com/` and the literature collected there

[7] Yvonne Choquet-Bruhat, James Isenberg, and James W. York, Jr.: *Einstein constraints on asymptotically Euclidean manifolds*, Phys. Rev. D 61, 084034 (2000)

[8] Piotr Chrusciel, Jalal Shatah: *Global existence of solutions of the Yang–Mills equations on globally hyperbolic four dimensional Lorentzian manifolds*, Asian J. Math. Vol. 1, No. 3, pp. 530 — 548, September 1997

[9] Ivan Costa e Silva, José Luis Flores, Jonatán Herrera: *Hausdorff closed limits and the c-boundary II: null infinity and black holes*, Class. Quantum Grav. 36 185007. arXiv:1807.00152

[10] Corvino J, Schoen RM: *On the asymptotics for the vacuum Einstein constraint equations.* J Differ Geom 73:185 —217 (2006)

[11] Helmut Friedrich: *Cauchy problems for the conformal vacuum field equations in general relativity*, Commun. Math. Phys. 91 (1983), 445 — 472.

[12] Greg Galloway: *Some rigidity results for spatially closed spacetimes*, Mathematics of gravitation, Part I (Warsaw, 1996), Banach Center Publ., vol. 41, Polish Acad. Sci., Warsaw, 1997, pp. 21 — 34 `http://matwbn.icm.edu.pl/ksiazki/bcp/bcp41/bcp4112.pdf`

[13] Claus Gerhardt: *Hypersurfaces of prescribed mean curvature in Lorentzian manifolds.* Mathematische Zeitschrift 235, pp 83 — 97 (2000)

[14] Sari Ghanem: *The global existence of Yang-Mills fields on curved space-times*, Journal of Hyperbolic Differential Equations 13(3):603-631. arXiv:1312.5476

[15] Steven Harris: *A triangle comparison theorem for Lorentz manifolds*. Indiana Univ. Math. J. 31, no. 3, 289308 (1982)

[16] Michael Kunzinger, Clemens Sämann: *Lorentzian length spaces*. Ann. Global Anal. Geom. 54, no. 3, 399 — 447 (2018). arXiv:1711.08990

[17] Olaf Müller: *Topologies on the future causal completion*. arXiv:1909.03797

[18] Olaf Müller: *Lorentzian Gromov-Hausdorff theory and finiteness results*. arXiv:1912.00988

[19] Barrett O'Neill: *Semi-Riemannian geometry: With applications to relativity*, Pure and Applied Mathematics, vol. 103, Academic Press, ISBN 0-12-526740-1 (1983)

[20] Barrett O'Neill: *The Geometry of Kerr Black Holes*, A K Peters, Ltd, 289 Linden Street, Wellesley, MA 02181, ISBN: 1-56881-019-9 (1995)

[21] Richard Schön, Shing-Tung Yau: *The Existence of a Black Hole Due to Condensation of Matter*, Commun. Math. Phys. 90, 575-579 (1983)