

A metric approach to Fréchet geometry

Olaf Müller*

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Abstract

The aim of this article is to present the category of bounded Fréchet manifolds in respect to which we will review the geometry of Fréchet manifolds with a stronger accent on its metric aspect. An inverse function theorem in the sense of Nash and Moser in this category is proved, and some applications to Riemannian geometry are given.

1 Introduction

The celebrated Nash-Moser inverse function theorem is one of the most powerful tools linking infinite-dimensional geometry with global analysis when dealing with smooth objects lacking a Banach structure. However, the usual description of the tame category uses the notion of a series of seminorms which is intuitively not very well-accessible, also it delivers not much geometric insight in these infinite-dimensional spaces (in most cases spaces of smooth sections). So an idea to remedy this gap could be to fix natural metrics on these spaces of sections which can encode the crucial properties of tameness. This will be done in the following article. Instead of tame maps (a notion that will refer to a metric instead of to a series of seminorms) we will often as well consider maps which are bounded w.r.t. the metrics used. For maps of finite differentiability this procedure would not be well-applicable as fundamental operators like covariant derivatives are unbounded, meanwhile in spaces of *smooth* maps with the usual metrics they *are* bounded. One reason to consider metrics instead of series of seminorms is certainly that they are geometrically more demonstrative. A second reason to be interested in concrete metrics on these spaces comes from applications in the geometric quantization of field theories. A third advantage consists in the existence of many fix point theorems for metric spaces. Thus the basic question of this article is: *How metric can an exhaustive description of the properties of Fréchet spaces be?*

The article, apart of presenting new results, is thought to serve as an introduction to Fréchet spaces from the metric point of view; in order to keep it as self-contained as possible, several well-known results (indicated as such) were included as well. It is structured as follows: In section 2 we review some basic results for connections in fiber bundles. The treatment of vector bundle connections as Levi-Civita connections on the total space does not seem to be mentioned in the literature so far. Likewise, some technical definitions and results about bounded geometry are established. In section 3, the category of bounded Fréchet manifolds is presented and applied to spaces of sections of fiber bundles. Finally, in section 4 we prove an inverse function theorem fitting to this category, right along the lines of the proof of the inverse function theorem for maps between Banach space. The last sections treats the question whether the notion of length structure is applicable in the present context.

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*Instituto de Matemáticas, Universidad Nacional Autónoma de México (UNAM) Campus Morelia, C. P. 58190, Morelia, Michoacán, Mexico. email: olaf@matmor.unam.mx

2 Preliminaries: Connections in fiber bundles and bounded geometry

2.1 General fiber bundles

As a preparation, we review some facts from the theory of connections.

Definition 2.1 Let a fibre bundle $\pi : N \rightarrow M$ be given, $m = \dim(M)$. A **connection on π** is a smooth m -dimensional horizontal distribution in τ_N , i.e. an m -dimensional subbundle D of τ_N with $\ker(d\pi|_D) = \{0\}$ at each point.

Definition 2.2 Let $\pi : N \rightarrow M$ be a fibre bundle with a connection D . Let a curve $c : [0, 1] \rightarrow M$ be given. Let $p \in \pi^{-1}(c(0))$ be given. Then a **horizontal lift of c starting at p** is the curve $L_{c,p}^D : [0, 1] \rightarrow N$ which is given by $L_{c,p}^D(0) := p$ and $\dot{L}_{c,p}^D(t) := (d\pi|_D)^{-1}(\dot{c}(t))$. By definition, $L_{c,p}^D$ projects to c under π .

Horizontal lifts are unique, but they do not exist always. If a connection gives rise to a horizontal lift for every curve in the base manifold, the connection is said to be **Ehresmann-complete** or an **Ehresmann-connection**.

It is easy to see by considering inverse curves that in the case of an Ehresmann connection each curve $c : [0, 1] \rightarrow M$ in the base manifold induces a diffeomorphism D_c from $\pi^{-1}(c(0))$ to $\pi^{-1}(c(1))$ which we call **parallel transport along c** . In the same manner, a curve of diffeomorphisms H_λ of M starting at the identity induces a curve of fiber bundle isomorphisms D_λ^H . If we have a metric fiber bundle with a connection for which the parallel transports along all curves are isometries of the fibers we call the connection **metric**.

Definition 2.3 (Pull-back of connections by immersions) Let $\pi : N \rightarrow M$ be a fiber bundle with a connection D . Let $f : S \rightarrow M$ be an immersion of a manifold S into M . Then we can equip the pull-back bundle $f^*\pi$ with the **pull-back connection** which is denoted by $f^*D := T(f^*N) \cap D$ where f^*N is understood as a submanifold of N .

It is easy to check that this is really a connection: First note that because of the finite-dimensional inverse function theorem around each point $p \in S$ there is a neighborhood U whose image is a submanifold of M . Then, by considering a locally trivial submanifold chart neighborhood of $f(U) \subset M$ we show that $f^*N = \pi^{-1}(f(U))$ is a submanifold. Then, for κ being the submanifold chart and for t being the trivialization, by considering the chart $(\kappa \times \mathbf{1}) \circ t$ for $\pi^{-1}(U)$ we see that $T(f^*N) = \{v \in TN \mid d\pi(v) \in T(f(U))\}$. As $\text{rang}(D) = m$ and $\text{corang}(T(f^*N)) = m - s$, we get $\text{rang}(f^*D) \geq s$. On the other hand, $d\pi(v) \in T(f(U))$ for $v \in T(f^*N)$, and $d\pi(v) \neq 0$ for $v \in (D \cap \pi^{-1}(m)) \setminus \{0\}$. Therefore $\ker(d(f^*\pi)|_{f^*D}) = \{0\}$, and the claim follows.

Proposition 2.4 Between the parallel transports of a connection and its pull-back connection under an immersion f we have the simple relation $f^*D_c = D_{f \circ c}$.

Definition 2.5 Let $\pi : N \rightarrow M$ be a fiber bundle with a connection D . A vector field X on N is called **projectable** iff $\pi_*(X(p)) = \pi_*(X(q))$ for all $p, q \in \pi^{-1}(m)$. It is called **basic** if it is projectable and horizontal.

Proposition 2.6 Let π be a fiber bundle with a connection. If X is a projectable vector field and Y is a vertical vector field, then $[X, Y]$ is again vertical.

Proof. This follows from $[X, Y] = \mathcal{L}_X Y = \frac{d}{dt}|_{t=0}(Fl_{-t}^X \circ Y \circ Fl_t^X)$ and from the fact that the local flow of X (which is a lift of the local flow of $\pi_* X$ and hence is defined on the whole fiber) translates fibers into fibers. \square

Let a fiber bundle $\pi : E \rightarrow M$ be given with a pseudo-Riemannian metric h on the fibres, a pseudo-Riemannian metric g on the base manifold M , and a connection D . Then we can use D to equip the total

space with the **total metric** $G = G_{g,h,D}$ which makes the projection a pseudo-Riemannian submersion. Then we can consider the Levi-Civita connection ∇^E on the total space. Now in the case that the metrics are Riemannian, the total Riemannian metric $G_{g,h,D}$ provides us with convenient criteria to determine whether a given connection is Ehresmann:

Theorem 2.7 *Let $\pi : E \rightarrow M$ be a bundle with a connection D . If, for some appropriately chosen Riemannian metrics g, h on M resp. on the fibers, $(E, G_{g,h,D})$ is complete, then D is Ehresmann. Likewise, if we can find complete metrics g and h such that the distortion of D_c as a map between the fibers is locally bounded with respect to the fiber metrics h , then D is Ehresmann.*

Proof in [4], 9.42 and 9.46 □

2.2 Vector bundle connections

For vector bundle connections we require additionally to the previous that they respect the linear structure:

Definition 2.8 (cf. [4], 9.51¹) *Let $\pi : E \rightarrow M$ be a vector bundle. A **vector bundle connection** on π is a connection on π as a fiber bundle whose parallel transports are linear maps of the corresponding fibers.*

Remark. In particular the definition above says that the parallel transport is defined overall, i.e. that the connection is Ehresmann.

A vector bundle connection gives rise to a covariant derivative in the following way: First, vector addition provides us with a map $\iota : T_p E \rightarrow \pi^{-1}(m)$ for all $p \in \pi^{-1}(m)$. Then for a vector $X \in T_p M$ and a section γ of $\pi : E \rightarrow M$ its covariant derivative $\nabla_X \gamma$ is the section of π defined by

$$\nabla_X \gamma(p) := \iota \left(\frac{d}{ds} (D_{c[s]}^{-1}(\gamma \circ c(s))) \Big|_{s=0} \right) = \iota \left(\lim_{s \rightarrow 0} \frac{D_{c[s]}^{-1}(\gamma(c(s))) - \gamma(c(0))}{s} \right) = \iota \left(\lim_{s \rightarrow 0} \frac{\gamma(c(s)) - D_{c[s]}(\gamma(c(0)))}{s} \right)$$

in which c is a curve with $c(0) = (p, t)$, $\dot{c}(0) = X$ and $c[s] := c|_{[0,s]}$. One immediately checks that this is independent of the choice of c .

Proposition 2.9 *Let D be a metric connection on the Pseudo-Riemannian vector bundle $(\pi : E \rightarrow M, g)$, let $f : S \rightarrow M$ be an embedding, then f^*D is metric on $(f^*\pi, f^*g)$.*

Proof by Proposition 2.4 □

We have a correspondance \sim (inverse to ι as above) between a point v in the total space and a vertical translational-invariant vector field \tilde{v} on the fiber. Translational-invariant vector fields are always vertical. Translational-invariant vector fields form a maximal center in the algebra of vector fields in the total space.

Proposition 2.10 *Let $\pi : N \rightarrow M$ be a vector bundle with vector bundle connection and arbitrary (not necessarily translational-invariant) metric on the fibers, let s be a section of π which is parallel along a vector field X on M , let \hat{X} the associated basic vector field to X , then $[\hat{X}, \tilde{s}]$ is translational-invariant.*

Proof. This is because the finitesimal commutator of the flow of \hat{X} (which is parallel transport along the flow of X and hence linear) and the flow of \tilde{s} (a translation in every fiber) is an affine map. □

In the following we examine the relation between a vector bundle connection and the Levi-Civita connection on the total space to the metric $G_{g,h,D}$ as in the previous section.

¹Note, however, that contrary to the claim of this otherwise excellent reference it is not sufficient that the structure group be *conjugate* to $Gl(n)$; it has to be $Gl(n)$ itself which is a distinguished subgroup of the diffeomorphisms in the fibre as soon as we have a vector bundle.

Proposition 2.11 *Let π be a fiber bundle with a Riemannian metric on the base space and the fibers and with a connection D as above, let ∇^E denote the Levi-Civita connection as above. Let X be a horizontal vector field and Y a vertical vector field. Then $\nabla_X^E Y$ is vertical.*

Proof. Consider one point p in the total space. Because of tensoriality, to calculate $\nabla_X Y$ at p , we can assume that X is projectable. Let V be another projectable vector field and show $\langle \nabla_X^E Y, V \rangle = 0$ by using Lemma 2.6. \square

Proposition 2.12 *Let a vector bundle π be given with a vector bundle connection and a translational-invariant metric. Let X be basic and V be translational-invariant. Then $\nabla_X V$ is translational-invariant.*

Proof. First recall that, as V is vertical and because of Proposition 2.11, $\nabla_X V$ is vertical. We show that it is invariant under *horizontal* translations, that means, under addition of vector fields of the form \tilde{Y} where Y is a horizontal section of π . Now let \tilde{Y} be as above and Z be another translational-invariant vector field, then

$$\begin{aligned} \tilde{Y}(\langle \nabla_X V, Z \rangle) &= \frac{1}{2} \tilde{Y}(X \langle V, Z \rangle - \langle X, [V, Z] \rangle + \langle Z, [X, V] \rangle + \langle V, [Z, X] \rangle) \\ &= \frac{1}{2} (X \tilde{Y} \langle V, Z \rangle - [X, \tilde{Y}] - 0 + \langle Z, [\tilde{Y}, [X, V]] \rangle + \langle V, [\tilde{Y}, [X, Z]] \rangle) \\ &= \frac{1}{2} (\langle Z, [\tilde{Y}, [X, V]] \rangle + \langle V, [\tilde{Y}, [X, Z]] \rangle) \\ &= \frac{1}{2} (\langle -Z, [X, -[V, \tilde{Y}]] - [V, [\tilde{Y}, X]] \rangle + \langle V, -[Z, [X, \tilde{Y}]] - [X, [\tilde{Y}, Z]] \rangle) \\ &= 0 \end{aligned}$$

where the first step is the Koszul formula, the second step is the definition of the Lie derivative resp. the translational invariance of the fiber metric, the third step uses Proposition 2.6, the fourth step is the Jacobi identity, and the last step uses Proposition 2.6 again and the fact that translations commute. \square

Proposition 2.13 *Let a vector bundle π be given with a metric vector bundle connection and a translational-invariant metric. Let \hat{X} be the basic vector field on E to a vector field X on M . Let Y be an X -parallel section of π , then $\nabla_{\hat{X}} \tilde{Y} = 0$.*

Proof. Choose another translational-invariant vector field \hat{Z} , w.r.o.g. $\nabla_X Z = 0$ at the point p in question, then the Koszul formula gives at p

$$\begin{aligned} \langle \nabla_{\hat{X}} \tilde{Y}, \hat{Z} \rangle &= \hat{X} \langle \tilde{Y}, \hat{Z} \rangle - \langle \hat{X}, [\tilde{Y}, \hat{Z}] \rangle + \langle \hat{Z}, [\hat{X}, \tilde{Y}] \rangle + \langle \tilde{Y}, [\hat{X}, \hat{Z}] \rangle \\ &= X \langle Y, Z \rangle + \langle \hat{Z}, \lim_{t \rightarrow 0} (P_t^X \circ t \tilde{Y} - t \tilde{Y} \circ P_t^X) \rangle + \langle \tilde{Y}, \lim_{t \rightarrow 0} (P_t^X \circ t \hat{Z} - t \hat{Z} \circ P_t^X) \rangle \\ &= \langle \hat{Z}, \lim_{t \rightarrow 0} \widetilde{P_t^X(Y)} \rangle + \langle \tilde{Y}, \lim_{t \rightarrow 0} \widetilde{P_t^X(Z)} \rangle = 0 \end{aligned}$$

where the second equation is due to the fact that $\langle \tilde{Y}, \hat{Z} \rangle$ constant on the fiber and the last equation is due to the fact that parallel transport of a metric connection is an orthogonal map and its derivative is a skew-symmetric map. \square

Now we can relate the connections by the following

Theorem 2.14 *Let a vector bundle π be given with a metric vector bundle connection and a translational-invariant metric. Then we have $\widetilde{\nabla_X^\pi v} = \nabla_X^E \tilde{v}$, where \hat{X} is the basic vector field corresponding to X .*

Proof. We have

$$\begin{aligned}
\widetilde{\nabla_X^\pi v} &= \lim_{t \rightarrow 0} \frac{P_X^t v - v \circ Fl_X^t}{t} \\
&= \lim_{t \rightarrow 0} \frac{\widetilde{P_X^t v} - v \circ \widetilde{Fl_X^t}}{t} \\
&= \lim_{t \rightarrow 0} \frac{P_{\hat{X}}^t \tilde{v} - \tilde{v} \circ Fl_{\hat{X}}^t}{t} \\
&= \nabla_{\hat{X}} \tilde{v}
\end{aligned}$$

where the first equation is a consequence continuity of the map $\tilde{\cdot}$, the second one of its linearity, the third one of Proposition 2.13. \square

Theorem 2.15 *Let a vector bundle π be given with a vector bundle connection and a translational-invariant metric. Then, if \tilde{w} is translational-invariant and v is vertical, $\nabla_v \tilde{w}$ is horizontal. If additionally the vector bundle connection is metric, then $\nabla_v \tilde{w} = 0$.*

Proof. Because of tensoriality we can assume that v is translational-invariant as well. Then the Koszul formula gives immediately $\langle \nabla_v \tilde{w}, z \rangle = 0$ for any vertical vector z which, again by tensoriality, we can assume to be the value of a translational-invariant vector field as well. On the other hand, the application of the same Koszul formula to a horizontal vector z gives

$$g(\nabla_v \tilde{w}, z) = \frac{1}{2}(-\mathcal{L}_z(g(v, \tilde{w})) + g(v, \mathcal{L}_z \tilde{w}) + g(\tilde{w}, \mathcal{L}_z v) = (\mathcal{L}_z g)(v, w)$$

which vanishes if the connection is metric, i.e. if the horizontal flow preserves the metric on the fibers. \square Furthermore, there is also a correspondence for basic vector fields:

Theorem 2.16 *Let a vector bundle $\pi : E \rightarrow M$ be given with a metric vector bundle connection and a translational-invariant metric. Then for any vector fields X, Y on M , for their horizontal lifts \hat{X}, \hat{Y} we have*

$$\nabla_{\hat{X}} \hat{Y} = \widehat{\nabla_X Y}.$$

Proof. This is because $\nabla_{\hat{V}}$ is horizontal:

$$\langle \nabla_{\hat{V}} \hat{W}, \hat{X} \rangle = \hat{V} \langle \hat{W}, \hat{X} \rangle - \langle \hat{W}, \nabla_{\hat{V}} \hat{X} \rangle = 0$$

and because of the corresponding formula controlling the horizontal part in general Riemannian submersions that can be found in [11], p. 212 (Lemma 45.3) which is proven using the Koszul formula. \square

Finally, the last possible combination in the covariant derivative vanishes:

Theorem 2.17 *Let a vector bundle $\pi : E \rightarrow M$ be given with a metric vector bundle connection and a translational-invariant metric. Then for any two vector fields X, V on M we have*

$$\nabla_{\hat{X}} \hat{V} = 0.$$

Proof. First show that $\nabla_{\hat{X}} \hat{V}$ is horizontal picking a translational-invariant vector field \tilde{Y} and using Theorem 2.15. On the other hand, $\nabla_{\hat{X}} \hat{V} = \nabla_{\hat{V}} \hat{X} - [\hat{V}, \hat{X}]$ must be translational-invariant following the theorems 2.15 and 2.10. This leaves us with the zero vector at every point. \square

So, for an arbitrary vector bundle $\pi : E \rightarrow M$ with translational-invariant metric and metric vector-bundle connection, and for any curve $c : \mathbb{I} \rightarrow E$ with $k := \pi \circ c$, $\dot{k} \neq 0$, we have

$$\dot{c}(t) = \dot{c}_{vert}(t) + \dot{c}_{hor}(t), \quad \dot{c}_{vert}(t) = \frac{1}{\|\dot{k}\|} \widetilde{\nabla_{k(t)}^{LC} c}(t) \quad (1)$$

while $\dot{c}_{hor} = \dot{k}$. This is because by definition $\nabla_{\dot{k}(t)}^{LC(E)} \tilde{c}(t) = 0$ if and only if \dot{c} is horizontal along \dot{k} by Theorem 2.14.

2.3 Bounded geometry

Definition 2.18 Let $F : \pi \rightarrow \pi'$ be a vector bundle homomorphism between two Riemannian vector bundles π, π' over manifolds M, N . The **upper resp. lower distortion $\bar{\delta}$ resp. $\underline{\delta}$** of F is defined as the quantities

$$\bar{\delta}(F) := \sup_{X \in SE} \|F(X)\| = \sup_{X \in E \setminus n} \frac{\|F(X)\|}{\|X\|},$$

$$\underline{\delta}(F) := \sup_{X \in SE} (\|F(X)\|^{-1}) = \sup_{X \in E \setminus n} \frac{\|X\|}{\|F(X)\|}$$

where $S\pi : SE \rightarrow M$ is the sphere bundle of π and n its zero section, and the **total distortion δ** of F as $\delta(F) := \max\{\bar{\delta}(F), \underline{\delta}(F)\}$.

Definition 2.19 Let (N, h) be a Riemannian manifold, let $V \subset M$ be open. We call (N, h) **of bounded geometry (in V)** if for the Levi-Civita connection on (N, h) holds: The injectivity radius ι_M is bounded from zero on M (resp. on V), and $\|\nabla^{(k)} R^M(p)\|_{(\tau_M^*)^3 \otimes \tau_M} \leq C_k$ for all $k \in \mathbb{N} \cup \{0\}$ and for all $p \in M$ (resp. $p \in V$). Let $\pi : E \rightarrow M$ be a fibre bundle $(E, h) \rightarrow (M, g)$ over a Riemannian manifold (M, g) with a fibre bundle connection D and a fiber metric. Let $U \subset E$ be open. We call π **of strongly bounded geometry (in U)** if the natural Riemannian metric on E is of bounded geometry (in U). We call it **of bounded geometry** if for every section γ there is an $\epsilon > 0$ such that for the ϵ -neighborhood U_ϵ of $\gamma(M) \subset E$, the bundle π is of strongly bounded geometry in U_ϵ .

Every bundle with compact fibers and compact base is of strongly bounded geometry, every bundle with compact base is of bounded geometry.

We can reformulate the conditions in the case of vector bundles:

Theorem 2.20 A vector bundle $\pi : E \rightarrow M$ is of bounded geometry if and only if the injectivity radius of the base is bounded from zero, if the curvature of the base satisfies $|(\nabla^{(M)})^{(k)} R^M| \leq C_k^{(M)}$ for all $k \in \mathbb{N} \cup \{0\}$, and if the vector bundle curvature satisfies $|(\nabla^{(\pi)})^{(k)} R^{(\pi)}| \leq C_k$ for all $k \in \mathbb{N} \cup \{0\}$.

Proof. First we treat the curvature conditions: The tensoriality of $\nabla^{(k)}$ and of R allows us to choose all horizontal fields basic and all vertical fields translation-invariant. Then we use the theorem 2.15, 2.14 and 2.16 to show that the bounds of $|\nabla^{(k)} R|$ in the definition above is equivalent to the existence of such bounds for the curvature terms of the corresponding vector bundle connections respectively bounds of the corresponding terms in the base. The geodesics split in two cases. Either $\dot{c}(0)$ is vertical; then c stays in the fibre and $\dot{c}(t) = \dot{c}(0)$ modulo translation. Or it has a horizontal part in which case the geodesic lies over the geodesic $k : \mathbb{I} \rightarrow M$ given by $k(0) = \pi(c(0))$ and $\dot{k}(0) = d\pi(\dot{c}(0))$, and one can easily check by means of Theorems 2.14 and 2.15 that the geodesic equations for c are equivalent to the ordinary differential system $\nabla_{k\iota}(\dot{c}^{vert}) = 0$ where ι is the usual isomorphism between the vertical space at a section and the total space of a bundle. One only needs to cover the image of k by convex neighborhoods and extend the vertical part of \dot{c} translational-invariant. Thus c is defined as long as k is defined. Therefore around $c(0)$ there is a cylindrical neighborhood $\pi^{-1}(B_R(\pi(c(0))))$ which contains a ball of radius R around $c(0)$. \square

There are some useful embedding theorems for sections of bundles of bounded geometry. For example, the compact supported smooth sections $\Gamma_0^\infty(\pi)$ are dense in the space of L_p -sections, for $1 \leq p < \infty$; and for

$s > k + \frac{n}{p}$ there is an embedding of $W^{s,p}$ sections into the space $C_b^k(\pi)$ of k times differentiable sections of finite C^k -norm, cf. [2].

Another useful property of bounded geometry is the following beautiful theorem which ensures the existence of a bounded atlas:

Theorem 2.21 *Let (M, g) be a Riemannian manifold of dimension n which is of bounded geometry. Then there is a ball B around 0 in the Euclidean space \mathbb{R}^n such that*

- (i) $\exp_p|_B$ is a diffeomorphism at every $p \in M$,
- (ii) The distortion $\delta(\exp_p|_B)$ is a bounded function on M ,
- (iii) There is a bound K_l independent of $p \in M$ such that all Christoffel symbols Γ_{ij}^k with respect to the normal neighborhood around all points $p \in M$ have C^i -norm smaller than K_l on $B \subset \mathbb{R}^n$.

For the **proof** cf. [12], Lemma (2.2) and Proposition (2.4) as well as the proof in [1], Appendix 2. \square

Theorem 2.22 *Let (M, g) and (N, h) be two Riemannian manifolds and $F : M \rightarrow N$ be a smooth map. Then for every compact subset $K \subset M$ there are constants $\kappa_{K,i}, \hat{\kappa}_{K,i}$ such that*

$$\langle (\nabla^N)^{i-1}(F_*V), (\nabla^N)^{i-1}(F_*V) \rangle_{(\tau_N^*)^i \otimes \tau_N} \leq \hat{\kappa}_{K,i} \cdot \langle (\nabla^M)^i V, (\nabla^M)^i V \rangle_{(\tau_M^*)^{(k)} \otimes \tau_N}$$

and

$$\begin{aligned} & \langle \nabla_{F_*X_1}^N \nabla_{F_*X_2}^N \dots \nabla_{F_*X_i}^N (F_*V), \nabla_{F_*X_1}^N \nabla_{F_*X_2}^N \dots \nabla_{F_*X_i}^N (F_*V) \rangle_{\tau_N} \\ & \leq \kappa_{K,i} \cdot \langle \nabla_{X_1}^M \nabla_{X_2}^M \dots \nabla_{X_i}^M V, \nabla_{X_1}^M \nabla_{X_2}^M \dots \nabla_{X_i}^M V \rangle_{\tau_M} \end{aligned}$$

for any vector fields $V, X_1 \dots X_i$ on K where the covariant derivatives are the Levi-Civita ones.

Proof. First we take a covering of every compact region K_n by coordinate neighborhoods as in Theorem (2.21). Then we pick arbitrary coordinates $(x_1 \dots x_{m-1}, t = x_m)$ at a given point $(p, \tau) \in M \times \mathbb{I}$. Then we show the second claim first: Show inductively that $\nabla_{X_1}^M \nabla_{X_2}^M \dots \nabla_{X_k}^M V$ is a sum of at most $k! \cdot 4^k \cdot m^{2k+1}$ terms of the form

$$f_{i,j_1^1 \dots j_{q(1)}^1} \cdot \dots \cdot f_{i,j_1^n \dots j_{q(n)}^n} \cdot \Gamma_{i_1 j_1, J_1^1 \dots J_{Q(1)}^1}^{k_1} \cdot \dots \cdot \Gamma_{i_p j_p, J_1^p \dots J_{Q(p)}^p}^{k_p} V_{j, k_1 \dots k_r} \partial_{x_n}$$

(where $X_i = \sum_j f_i^j \partial_j$ with $q(i), Q(i), p, r \leq n$; this can be done by the observation that each factor in front of ∂_{x_n} can be derived producing m terms of one factor more and ∂_{x_n} can be derived producing m^2 terms of two factors more). Then consider the pushed-forward coordinate system and apply the coordinate formula for the Christoffel symbols in partial derivatives of the metric. The partial derivatives of the coefficients f_i^j stay the same on both sides.

For the first part of the claim note that

$$\nabla^{n+1}(X_1, \dots X_{n+1})V = \nabla_{X_{n+1}}(\nabla^n(X_1, \dots X_n)V) - \sum_{i=1}^n \nabla^n(X_1, \dots \nabla_{X_{n+1}} X_i \dots X_n)V$$

and show, again by induction, that $\nabla^n(X_1, \dots X_n)V$ is a sum of $n!$ terms of the form

$$\nabla_{\nabla_{V(1,1)} \dots \nabla_{V(I(1),1)} V(1)} \dots \nabla_{\nabla_{V(1,m)} \dots \nabla_{V(I(m),m)} V(m)} V$$

where $V(i, j), V(l) \in \{X_1, \dots X_n\}$ and if k denotes the index we have

$k(V(i, j)) > k(V(i+1, j)) > k(V(j)) > k(V(j+1))$. Thus $\hat{\kappa}_k := (n! \max_{i \in 0, \dots, k} \kappa_i)^2$ does the job. \square

3 Spaces of smooth sections and Fréchet geometry

In this section we provide the foundations of Fréchet spaces and Fréchet manifolds, thus the material presented is novel only when dealing with concrete metrics. The non-metric aspects might be found in [8] or in [3], however, we will include some of their proofs here.

3.1 Fréchet spaces

A motivation for generalizing the notion of Banach spaces is the fact that there is no Banach space structure on the space of smooth sections of a vector bundle such that the covariant derivative in the direction of any nonzero vector field on the base be continuous. This is seen by noting that in Banach spaces continuity is equivalent to boundedness, and by constructing sections which are arbitrarily high eigenvalues of the derivative or its square (most easily seen in $C^\infty(\mathbb{S}^1, \mathbb{R})$ with $f_K(x) = K^{-1} \sin(K^2 x)$).

Definition 3.1 A **Fréchet space** is a locally convex topological vector space V whose topology can be induced by a complete translational-invariant metric d on V . If we keep track of its metric we call the pair (V, d) **metric Fréchet space**. For a real number K , a Fréchet space is called **scalar-bounded by K** iff $d(\rho \cdot v, 0) \leq K \rho d(v, 0)$ for every $\rho \geq 1$.

Remark. The triangle inequality implies that every Fréchet space with star-shaped balls is scalar-bounded by 2. However, even in finite-dimensional metric vector spaces, balls do not have to be star-shaped. As an example, consider the real line with the metric $d(r, s) := \Phi(|r - s|)$ with $\Phi(x) := x$ for $0 \leq x \leq 1$, $\Phi(x) := 1 - (x - 1)/2$ for $1 \leq x \leq 2$ and $\Phi(x) := 1/2 + (x - 2)/3$ for $x \geq 2$. as $\Phi(x \pm y) \leq \Phi(x) + \Phi(y)$, the metric d satisfies the triangle inequality, but the balls with radius $1/2 \leq r \leq 1$ are not starshaped and not even connected in this example.

Example 1: Every Banach space $(V, \|\cdot\|)$ is a metric Fréchet space scalar-bounded by 1, e.g. finite-dimensional vector spaces, $W^{k,p}$ and C^p , with $d(v, w) := \|v - w\|$.

Example 2: The vector space $F^\mathbb{N}$ of sequences in a fixed metric Fréchet space (F, d) can be made a metric Fréchet space $(F^\mathbb{N}, D)$ scalar-bounded by 1 by setting

$$\delta_n((v_i)_{i \in \mathbb{N}}, (w_i)_{i \in \mathbb{N}}) := \sum_{i=1}^n d(v_i, w_i)$$

picking, for a $D > 0$, a concave monotonously increasing continuous function $\Phi : [0, \infty)$ with $\Phi(0) = 0$ (from now on we will take the special choice $\Phi : [0, \infty) \rightarrow [0, 1)$, $\Phi(x) = x/(1 - x)$), a positive sequence α converging to 0, and defining the **standard α -metric**

$$D_\alpha((v_i)_{i \in \mathbb{N}}, (w_i)_{i \in \mathbb{N}}) := \sum_{n \in \mathbb{N}} \alpha_n \Phi(\delta_n((v_i)_{i \in \mathbb{N}}, (w_i)_{i \in \mathbb{N}}))$$

as well as the **supremum α -metric**

$$d_\alpha((v_i)_{i \in \mathbb{N}}, (w_i)_{i \in \mathbb{N}}) := \sup_{n \in \mathbb{N}} \alpha_n \cdot \Phi(\delta_n((v_i)_{i \in \mathbb{N}}, (w_i)_{i \in \mathbb{N}})).$$

Each of these choices defines a natural family of metrics for the adequate category of Riemannian fiber bundles with connection. The supremum metrics have the advantage to have convex balls while they are nowhere differentiable. In contrast, as Φ is differentiable on $[0, \infty)$, the standard metrics have better differentiability properties but none of their balls is convex. An interesting open question is whether there is a metric with both advantages. In contrast to Glöckner ([7]) we choose the name *standard metric* for the *first* one as the usual metric appearing in the literature is D_α for $\alpha_n := 2^n$. For $r > 0$ let $l(r)$ be the sequence with $l(r)_n = r^n$, then we have the estimate

$$D_{l(r^2)} < d_{l(r)} < D_{l(r)} \quad (2)$$

which allows us to translate results obtained in the supremum metrics into the context of standard metrics and vice versa. The second one of these inequalities is obvious, for the first one note that as $l(r)$ tends to zero and the range of Φ is bounded, the supremum s is attained by, say, $r^n \Phi(a_n)$. Then for all $j \in \mathbb{N}$ we have $\Phi(a_j) \leq r^{-n+j} s$ and therefore

$$D_{l(r^2)}(a) = \sum_{i=1}^{\infty} r^{-2i} \Phi(a_i) \leq \sum_{i=1}^{\infty} r^{-2i} r^{-n+i} s \leq r^{-n} s \sum_{i=1}^{\infty} r^{-i} = r^{-n} s.$$

Note, however, that the inverse estimates $D_{l(r)} \leq C d_{l(r)}$ or $d_{l(r)} \leq C D_{l(r^2)}$ that would complete (2) to equivalences between metrics, are wrong.

Example 3:

Let (M, g) be a Riemannian manifold and $\pi : E \rightarrow M$ a vector bundle over (M, g) equipped with a Riemannian vector bundle metric $\langle \cdot, \cdot \rangle$ and a metric covariant derivative. Now we define metrics on $\Gamma_p(\pi)$ by pull-back of the above standard and Glöckner metrics applied on the series of L^p norms in the case of a compact base (by convention, put $L^\infty = C^0$):

$$\|\gamma\|^{(p,n)} := \|\sqrt{\langle \nabla^{(n)} \gamma, \nabla^{(n)} \gamma \rangle_{(\tau_M^*)^{\otimes n} \otimes \pi}}\|_{L^p(M)}$$

$$\|\gamma\|_{p,n} := \sum_{i=1}^n \|\gamma\|^{(p,i)}$$

$$\langle \gamma \rangle_{p,\alpha} := D_\alpha((\|\gamma\|_{p,n})_{n \in \mathbb{N}}, 0), \quad (3)$$

$$\langle \gamma \rangle_{p,\alpha}^s := d_\alpha((\|\gamma\|_{p,n})_{n \in \mathbb{N}}, 0) \quad (4)$$

giving rise to $D_{p,\alpha}$ and $d_{p,\alpha}$, respectively, by applying $\langle \gamma \rangle_{p,\alpha}$ and $\langle \gamma \rangle_{p,\alpha}^s$ to differences. Likewise, if M is noncompact, we define in a similar way standard and supremum metrics on $\Gamma(\pi)$ using an increasing series $C_n \subset C_{n+1}$ of compacta whose union is M . As we assume M to be Riemannian and complete, we restrict ourselves to the choice $C_n := \overline{B}_n(x)$ where x is an arbitrarily fixed point in M . Then we define

$$d_{x,p,\alpha}(\gamma, \delta) := \sum_{i=1}^{\infty} 2^{-i} d_{x,p,\alpha}^i(\gamma, \delta) \quad , \quad d_{x,p,\alpha}^i(\gamma, \delta) := d_\alpha\{|\gamma - \delta|_{K_j}\|_{W^{i,p}}\}_{j \in \mathbb{N}}, 0, \quad (5)$$

$$D_{x,p,\alpha}(\gamma, \delta) := \sum_{i=1}^{\infty} 2^{-i} D_{x,p,\alpha}^i(\gamma, \delta) \quad , \quad D_{x,p,\alpha}^i(\gamma, \delta) := D_\alpha\{|\gamma - \delta|_{K_j}\|_{W^{i,p}}\}_{j \in \mathbb{N}}, 0. \quad (6)$$

The estimates (2) immediately carry over to this case.

Remark. We could have made Definition (3) instead of (5) also in the case of M noncompact, if we assume that π bounded and if we restrict ourselves to the spaces of sections all of whose Sobolev norms are finite. These are metric Fréchet spaces as well, natural in the class of (bundles over) Riemannian manifolds and isometries *without* base points; but as the restriction to these subspaces has some disadvantages we stick to (5) in the noncompact case.

Theorem 3.2 *Let $\pi : E \rightarrow M$ be a vector bundle, then the space of its smooth sections $\Gamma_p(\pi)$ with the metrics $d_{p,\alpha}$ or $D_{p,\alpha}$ (in the compact case) resp. $d_{x,p,\alpha}$ or $D_{x,p,\alpha}$ is a metric Fréchet space scalar-bounded by 1. The resulting topology is the same for all these metrics and is finer than the compact-open topology, it equals the inverse limit topology of the Banach spaces of $W^{k,p}$ sections in the compact case. The supremum metrics have convex balls, while none of the balls of the standard metrics is convex.*

Proof. It is enough to show that the metrics (i) are translation-invariant, (ii) are complete and has (iii) the balls in the supremum metrics are convex (then the topology is locally convex in either case).

Property (i) is clear by definition. For (iii), we have to show that for each two sections γ, γ' we have $\langle \gamma + t(\gamma' - \gamma) \rangle_{p,\alpha}^s \leq \max\{\langle \gamma \rangle_{p,\alpha}^s, \langle \gamma' \rangle_{p,\alpha}^s\}$ for all $t \in [0, 1]$. By continuity of the metric it is sufficient to restrict ourselves to the case $t = 1/2$, and the rest is then done by nesting by intervals. By concavity of the arithmetic sum and monotonicity of Φ we have

$$\Phi\left(\frac{a_n + b_n}{2}\right) \leq \max\{\Phi(a_n), \Phi(b_n)\}$$

and therefore, for $d_{\alpha,p}^s$,

$$\alpha_n \Phi\left(\frac{a_n + b_n}{2}\right) \leq \max\{\Phi(a_n), \alpha_n \Phi(b_n)\}$$

and thus

$$\sup(\alpha_n \Phi\left(\frac{a_n + b_n}{2}\right)) \leq \max\{\sup(\alpha_n \Phi(a_n)), \sup(\alpha_n \Phi(b_n))\}.$$

For Property (ii), consider a Cauchy sequence of smooth sections. Then we have to show that there is a limit in $\Gamma(\pi)$. But

$$\langle \delta \rangle \geq \sum_{n=0}^k \Phi(\alpha_n \|\delta\|_{n,p}) \cdot 2^{-n} \geq 2^{-k} \sum_{n=0}^k \Phi(\alpha_0 \|\delta\|_{n,p})$$

Therefore a Cauchy sequence γ_n in $\Gamma(\pi)$ is a Cauchy sequence in the Banach space $W^{k,2}$ as well (put $\delta = \gamma_k - \gamma_l$ and use the continuity of Φ at 0 and the invertibility of Φ in the positive semiaxis) and has therefore a limit γ in this Banach space. Because of uniqueness this limit is the same in all these Banach spaces. As we can use the Sobolev embedding theorems in every C_n , it is smooth. For the statement about the compact-open topology note that the form of the standard metric shows that a set A is open in the Fréchet space if it is open in every $\Gamma_p^k(\pi)$. The metrics $D_{p,\alpha}$ and $D_{x,p,\alpha}$ are scalar-bounded by 1 as every of its additive terms is. The metrics $d_{p,\alpha}$ and $d_{x,p,\alpha}$ are scalar-bounded by 1 as well because if for fixed $v \in \Gamma(\pi)$ we consider the real function $L : t \mapsto d_{p,\alpha}(tv, 0)$. As this function consists piecewise of concave functions which vanish in 0, it is easy to see that $L(s \cdot t) \leq sL(t)$. \square

In all the examples the metric could be constructed by a countable family of seminorms. This is a general feature of Fréchet spaces as shown by the following definition and the following theorem that can be found in [8], [3]:

Definition 3.3 A seminorm $\|\cdot\|$ on a vector space F is a real-valued, nonnegative function which is subadditive and satisfies $\|cf\| = |c| \cdot \|f\|$ for all scalars c and vectors f .

Theorem 3.4 (cf. [8], eg.) Let F be a Fréchet space. Then there is a \mathbb{N} -family of continuous seminorms $\|\cdot\|_i$ on F whose balls $B_\epsilon^i(x) := \{y \in X : \|y - x\|_i < \epsilon\}$ are a basis of the topology of F . Therefore the topology of F can be generated by the metric

$$D_\alpha(f, g) := \sum_{i=1}^{\infty} \alpha_i \Phi(\|f - g\|_i) \tag{7}$$

where α is an arbitrary positive sequence converging to 0, and as well by

$$d_\alpha(f, g) := \sup_{i \in \mathbb{N}} \alpha_i \Phi(\|f - g\|_i). \tag{8}$$

Proof. Choose a Fréchet metric d , consider $B_{\frac{1}{\lambda}}^d(0)$ and define the seminorms as the so-called Minkowski functionals

$$\|v\|_i := \inf\{\lambda > 0 \mid \frac{1}{\lambda} \cdot v \in U_i\}.$$

where we choose convex subsets $U_i \subset B_{\frac{1}{\lambda}}^d(0)$. These Minkowski functionals are subadditive, as for $\frac{1}{\lambda}f, \frac{1}{\mu}g \in U_i$ we have also $\frac{1}{\lambda+\mu}(f+g) \in U_i$ as a convex combination. Continuity is an easy consequence of subadditivity. Finally, Cauchy sequences w.r.t. all $\|\cdot\|_i$ are Cauchy sequences for the metric. \square

It should be stated, however, that if the metric d was already given as a sum of seminorms as in Equation (7), the Minkowski functionals will *not* give us back the original seminorms (although the tame equivalence class stays the same as we will see in Theorem 3.6). One reason why the description by seminorms is important is that it appears in the Nash-Moser inverse function theorem which is valid only in the case of *tame* Fréchet spaces and *tame* operators which in turn are defined in terms of the seminorms. Thus let us recall some definitions from the tame category:

Definition 3.5 A linear map A between Fréchet spaces F, G with sequences of seminorms $\|\cdot\|_i$ and $\|\cdot\|'_i$, respectively, is called **tame (from $(F, \|\cdot\|_i)$ to $(G, \|\cdot\|'_i)$)** if there are natural numbers b, r such that A satisfies estimates $\|v\|'_n < C_n \|v\|_{n+r}$ for $n > b$. Two different sequences of seminorms $\|\cdot\|$ and $\|\cdot\|'$ on one and the same Fréchet space F are called **tamely equivalent** if there is a linear isomorphism $A : F \rightarrow F$ which is tame from $(F, \|\cdot\|_i)$ to $(F, \|\cdot\|'_i)$ and whose inverse is tame from $(F, \|\cdot\|'_i)$ to $(F, \|\cdot\|_i)$.

Theorem 3.6 Let F be a Fréchet space with a metric defined by Equation 7 or Equation 8. Then the sequence of norms defining the metric is tamely equivalent to the sequence of 2^i -Minkowski functionals.

This theorem (whose straightforward proof shall be skipped) may sound quite technical, but it implies that tameness of maps does not depend on the choice of a special sequence of seminorms but only on the metric as the latter one gives rise to the sequence of Minkowski functionals. Thus tameness is an inherent notion of the *metric* category and will be related to the sequence of 2^i -Minkowski functionals from now on. It even only depends on the equivalence class of a metric which is shown by the following theorem whose proof will be omitted as well:

Theorem 3.7 Let $A : (F, d) \rightarrow (G, d')$ be bounded by C , then A is tame with grade $r \in \mathbb{N}$ whenever $2^r > C$, base 0 and $C_n = 1$ for all $n \in \mathbb{N}$.

Let us come back to our examples. Comparing Example 2 and Example 3 we notice that in Example 2 none of the seminorms we used is a norm while in Example 3 any of the seminorms is a norm. The question could arise whether there is *any* continuous norm on $F^{\mathbb{N}}$. This question is answered negatively in the following theorem.

Theorem 3.8 (cf. [8]) The Fréchet space $F^{\mathbb{N}}$ does not have a continuous norm.

Proof. Let us assume the existence of a continuous norm ν . Then we consider a ball $B_R^\nu(0)$. On one hand, this ball cannot contain any nontrivial subspace of $F^{\mathbb{N}}$, as ν is homogeneous w.r.t. the multiplication by positive numbers. But on the other hand, the ball is open because of continuity of ν , so it contains a finite intersection of elements $B_R^{\|\cdot\|_i}(0)$ of the basis of the topology. But an intersection of the balls for the seminorms $\|\cdot\|_{i_1}, \dots, \|\cdot\|_{i_n}$ contains the subspace $\{x \in F^{\mathbb{N}} \mid x_1 = \dots = x_m = 0\}$ where $m = \max\{i_1, \dots, i_n\}$, a contradiction. \square

Example 4: Spaces of maps $(C^k, W^{k,p}$ or smoothly finite) from a Riemannian manifold to a metric Fréchet space F can be made metric Fréchet spaces (with the same scalar bound). This example can be generalized to sections of Fréchet vector bundles.

A theorem which provides us with still more examples for metric Fréchet spaces (and whose non-metric variant can be found in [8] or [3]) is:

Theorem 3.9 (1) A closed subspace of a metric Fréchet space is again a metric Fréchet space, scalar-bounded by the same constant.

(2) A quotient of a metric Fréchet space by a closed subspace is again a metric Fréchet space, scalar-bounded by the same constant.

(3) The direct sum of finitely many metric Fréchet spaces is again a metric Fréchet space, scalar-bounded by the maximum of the bounds.

Proof. (i) Restrict the metric to the subspace and consider the relative topology of the closed subspace. Convex sets stay convex as intersected with a linear subspace. The scalar bound is trivial.

(ii) Let us call the closed subspace U and the surrounding Fréchet space X . Define the new metric d' by $d'(v, w) := \min_{c \in U} d(v + c, w) = \min_{c, d \in U} d(v + c, w + d)$ (the last equation is valid because of the invariance of d under translations). This metric generates the quotient topology. Now for every Cauchy sequence in X/U we have to find a Cauchy sequence of representatives in X . Thus choose a $M_\epsilon \in \mathbb{N}$ s.t. for all $m, n > M$ we have $d'([v_m], [v_n]) = \min_{c \in U} d(v_m, v_n + c) < \frac{\epsilon}{3}$. Then choose a $m(0) > M$, a representative $v_{m(0)}$ and a sequence of vectors $c_n \in U$ with $d(v_{m(0)}, v_n + c_n) < \frac{\epsilon}{2}$. Then using the triangle inequality we see that for $\tilde{v}_n := v_n + c_n$ we have $d(\tilde{v}_k, \tilde{v}_l) < \epsilon$. Now modify the sequence of representatives successively this way for $\epsilon = \frac{1}{n}$ for all $n \in \mathbb{N}$. this converges and leaves us with a Cauchy sequence in X . For the scalar bound and for $\rho \geq 1$ take $\tilde{c} := \rho \cdot c$ in the definition of the distance.

(iii) Let d_1, d_2 the two metrics, we choose a continuous concave function $\Delta : \mathbb{R}^2 \rightarrow \mathbb{R}$ (e.g. $x_1 + x_2$ or $\sqrt{x_1^2 + x_2^2}$) and define the new metric $d' := \Delta \circ (d_1, d_2)$. For the scalar bound use concavity of Δ . \square

Theorem 3.10 (metric Hahn-Banach-theorem) Let F be a Fréchet space, $G \subset F$ a subspace and $\lambda : G \rightarrow \mathbb{R}$ a continuous linear map. Then there is a continuation of λ to a continuous linear map $F \rightarrow \mathbb{R}$. If we fix a Fréchet metric d with respect to which λ is bounded on G by R , we can choose a continuation bounded by R as well. In particular, for every vector $f \in F$, there is a continuous linear functional λ on F with $\lambda(f) \neq 0$.

Proof. The proof is in complete analogy to the Banach case, cf. [13], pp. 372-378. For the question of boundedness take the Theorem of Hahn-Banach as quoted in cf. [13], pp.94-97: Let X be a real vector space, U a subvector space of X , $p : X \rightarrow \mathbb{R}$ sublinear, $l : U \rightarrow \mathbb{R}$ linear with $l(u) \leq p(u)$, for all $u \in U$. Then there is a linear continuation $L : X \rightarrow \mathbb{R}$, $L|_U = l$, with $L(x) \leq p(x)$, for all $x \in X$. To apply this, take $p(x) = \sup_{u \in U} \frac{|l(u)|}{d(u,0)} \cdot d(x,0)$. \square

Theorem 3.11 (Open-mapping-theorem) Let F, G be Fréchet spaces. If $T : F \rightarrow G$ is a linear map which is continuous and surjective, then T is an open map. In particular, if T is continuous and bijective, T is a homeomorphism.

Proof analogously to the Banach case, cf. [15], p.75 \square

Definition 3.12 A subspace G of a Fréchet space F is called **topologically complemented in F** if there is another subspace H of F such that the map $\Phi : G \times H \rightarrow F, \Phi((g, h)) := g + h$ is a homeomorphism, i.e. if F is homeomorphic to the topological direct sum $G \oplus H$. In this case we call H a **topological complement of G in F** .

As there is a homeomorphism from G to $G \oplus \{0\} \subset G \oplus H$, a topological complemented subspace is always closed. Likewise, it is easy to see that G is topologically complemented in F if and only if there is a continuous projection $\pi : F \rightarrow G$.

Example: Hamilton ([8]) gives an example of a closed subspace of a Fréchet space which is not topologically complemented. So take $F := C^\infty([0, 1])$ which contains the space G of 1-periodic real functions on the real line, $C_1^\infty(\mathbb{R})$ by the restriction ρ on the unit interval. If we define

$$p : C^\infty([0, 1]) \rightarrow \mathbb{R}^{\mathbb{N}}, f \mapsto D^j f(2\pi) - D^j f(0)$$

we get the short exact sequence

$$\{0\} \rightarrow C_1^\infty(\mathbb{R}) \xrightarrow{\rho} C^\infty([0, 1]) \xrightarrow{p} \mathbb{R}^{\mathbb{N}} \rightarrow \{0\}$$

Thus the quotient of $C^\infty([0, 1])$ by $C_1^\infty(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$. As the latter one does not have any continuous norm, there cannot be a continuous linear isomorphism between $\mathbb{R}^{\mathbb{N}}$ and any closed subspace of F . Therefore the above sequence does not split, and G is not topologically complemented in F .

This behaviour is not exceptional which is shown by the following theorem:

Theorem 3.13 (cf. [10], p. 435) *Let F be a Fréchet space with a continuous norm which is not Banach. Then there is a closed subspace $H \subset F$ with $F/H \cong \mathbb{R}^{\mathbb{N}}$, thus H is not topologically complemented in F .*

But at least simple subspaces of Fréchet spaces are topologically complemented:

Theorem 3.14 ([10]) *Let F be a Fréchet space. Then*

- (1) *Every finite-dimensional subspace of F is closed.*
- (2) *Every closed subspace $G \subset F$ with $\text{codim}(G) = \dim(F/G) < \infty$ is topologically complemented in F .*
- (3) *Every finite-dimensional subspace of F is topologically complemented.*
- (4) *Every linear isomorphism between the direct sum of two closed subspaces and F , $G \oplus H \rightarrow F$, is a homeomorphism.*

3.2 Differentiation and integration of Fréchet maps

Contrary to the case of Banach spaces we will have several different notions of differentiability of maps between Fréchet spaces. First define the spaces $L(F, G)$ of set-theoretic linear maps and $CL(F, G)$ of continuous linear maps between Fréchet spaces F, G . Then we note that $CL(F, G)$ can be made a topological vector space by the compact-open topology.

Theorem 3.15 *Let F, G be topological vector spaces, let G be locally convex and metrizable. Then the set $CL(F, G)$ of continuous linear maps topologized by the compact-open topology is a topological vector space.*

Proof. Choose a metric d inducing the topology of G . Let $A+B = C$ in $CL(F, G)$, let $C \subset (K, O)$. Then for $M := A(K), N := B(K)$ we know that M, N and therefore also $M + N$ compact and $M + N \subset O$. Now let $\epsilon := d(M+N, \partial O) > 0$. Then $B_{\frac{\epsilon}{2}}(M) + B_{\frac{\epsilon}{2}}(N) \subset O$ by the triangle inequality, and $(K, B_{\frac{\epsilon}{2}}(M)) \times (K, B_{\frac{\epsilon}{2}}(N))$ is an open neighborhood of (A, B) which is mapped in (K, O) under $+$. Thus $+$ is continuous. Now let us prove that the scalar multiplication is continuous as well. Let $r \in \mathbb{R}, A \in CL(F, G)$ be given as well as an open neighborhood of rA of the form (K, O) . We have to find a positive number s and an open neighborhood U of A with $(r-s, r+s) \times U$ mapped into (K, O) by scalar multiplication. First note that there is an $\epsilon > 0$ with $B_\epsilon(rA(K)) \subset O$. Now consider the continuous function $R : B \mapsto \sup_{x \in K} d(A(x), B(x))$ on $CL(F, G)$ and define $U := R^{-1}([0, \frac{\epsilon}{8r}))$. Choose a symmetric convex open neighborhood W of 0 contained in $B_{\epsilon/2}(0)$ and consider the continuous Minkowski functional $\rho(v) := \inf\{t \in \mathbb{R} \mid \pm t^{-1}v \in W\}$ on the compact set AK where it attains a finite maximum which we call s' . Put $s := \min\{s', r\}$. Now $(r-s, r+s) \times U$ has the desired property: Let $(\tau, B) \in (r-s, r+s) \times U$, let $k \in K$, then

$$\begin{aligned} d(\tau Bk, rAk) &\leq d(\tau Bk, \tau Ak) + d(\tau Ak, rAk) \\ &\leq S\tau d(Bk, Ak) + d((\tau-r)Ak, 0) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

where S is a bound of scalar multiplication which can be chosen smaller or equal to 2. This concludes the proof. \square

At a first glance, this looks good: Why not define differentiability between Fréchet spaces by means of the spaces $CL(F, G)$? The overnext theorem will tell us that this concept would not be very far-reaching as it does not allow for the iterative definition of higher differentiability. As a preparation for its proof we define certain subsets of $\Gamma(\pi)$ which will turn out to be compact and generic for compact subsets in the sense of the next theorem. For $\gamma \in \Gamma(\pi)$ and an $a \in \mathbb{R}^{\mathbb{N}}$ we set

$$K^{\gamma, \alpha} = \{s \in \Gamma(\pi) \mid \|\beta - \gamma\|_n \leq a_n \text{ for all } \beta \in B \text{ and all } n \in \mathbb{N}\}.$$

As an intersection of closed sets, every $K^{\gamma, \alpha}$ is closed. The following theorem shows that it is compact as well.

Theorem 3.16 *A closed set $B \subset \Gamma(\pi)$ is compact if and only if there is a $\gamma \in \Gamma(\pi)$ and an $a \in \mathbb{R}^{\mathbb{N}}$ with $B \subset K^{\gamma, \alpha}$.*

Proof. Assume the condition fails, then there is an $n \in \mathbb{N}$ such that B is unbounded w.r.t. $\|\cdot\|_n$ and, consequently, contains a sequence β_n which in turn does not contain any Cauchy subsequences in this norm. Therefore none of its subsequences converges in $\|\cdot\|_n$ and thus neither in the Fréchet space $\Gamma(\pi)$. On the other hand, it is easy to check that each $K^{\gamma, \alpha}$ is closed in $\Gamma(\pi)$. Now, for a given $K^{\gamma, \alpha}$, define $K_l^{\gamma, \alpha} := \{\delta \in \Gamma^{l+1} \mid \|\delta - \gamma\|_i \leq \alpha_i \forall i = 1, \dots, l+1\}$. The closure $\overline{K_l^{\gamma, \alpha}}$ of $K_l^{\gamma, \alpha}$ in $\Gamma^l(\pi)$ is compact because of the Arzela-Ascoli Theorem. The series $\overline{K^{\gamma, \alpha}}$ converges to $\overline{K^{\gamma, \alpha}} = K^{\gamma, \alpha}$ in the inverse limit $\Gamma(\pi) = \lim_{\leftarrow} \Gamma^l(\pi)$, thus by the compatibility of compactness of Hausdorff spaces and inverse limits the latter one is compact in $\Gamma(\pi)$ (this fact can be looked up in [6] or [5], App.2, 2.4., or seen by the fact that the inverse limit is defined as a subset of the Tychonoff product which is compact in this case and it is a closed subset by the constituents of the limit being Hausdorff). \square

Theorem 3.17 *$CL(F, G)$ is not metrizable.*

Proof. Recall that every metric space is first-countable, i.e. every point has a countable neighborhood base. Now take $0 \in CL(F, G)$ and assume that 0 has a countable neighborhood base U_n . Every U_n is the union of finite intersections of sets of the form (K_i, O_i) where the O_i are open neighborhoods of $0 \in G$. So for every n we pick one finite intersection of the union and put $\tilde{U}_n := (\bigcup K_i, \bigcap O_i)$. Note that $\tilde{U}_n \neq \bigcap O_i$ as $\bigcap O_i$ is an open neighborhood of 0. If $\{U_n\}_{n \in \mathbb{N}}$ was a neighborhood base so is $\{\tilde{U}_n\}_{n \in \mathbb{N}}$ as $\tilde{U}_n \subset U_n$ for every n . Now put $U'_n := (K^{\gamma_n, \alpha_n}, \bigcap O_i)$ where K^{γ_i, α_i} contains $\bigcup K_i$. W.r.o.g. we can choose $\gamma_i = 0$ for all i . Then, following Cantor's diagonal procedure, define $\omega_k := 2(\alpha_k)_k$ and $K := K^{\gamma, \omega}$. This is a neighborhood of 0 which contains none of the system above. This concludes the proof by contradiction. \square

Definition 3.18 *Let F, G be two Fréchet spaces, $U \in F$ open. A map $Q : U \rightarrow G$ is called **differentiable at $p \in U$** iff there is a linear map $A_p : F \rightarrow G$ with*

$$A_p(v) = \lim_{t \rightarrow 0} \frac{Q(p+tv) - Q(p)}{t}$$

*for all $v \in F$. If Q is differentiable at all points $p \in U$, if $dQ(P)$ is continuous for all $p \in U$ and if the induced map $dQ : U \rightarrow CL(F, G)$ is continuous, we call Q **c-differentiable**. The space of all c-differentiable maps from U to G is denoted by $C^1(U, G)$.*

Theorem 3.19 *For an open set $U \in F$ a map $Q : U \rightarrow G$ is c-differentiable iff it is differentiable at every point and if the map $Q' : U \times F \rightarrow G, (u, f) \mapsto dQ(u)f$ is continuous in the product topology.*

Proof. Continuity of df at every point is trivial. Let $df : U \rightarrow CL(F, G)$ be given, and without restriction of generality let $df(p) = 0$. Let an element of the basis $V = (K, O) \ni 0$ be given. We have to show that there is an open neighborhood U of p with $df(U) \subset V$ or equivalently $f'(u, K) \subset O$ for all $u \in U$. By continuity

of f' , around every point $q \in K$ we can find an open neighborhood W_q of q and an open neighborhood U_q of p with $f'(U_q \times W_q) \subset O$. Let W_1, \dots, W_n be a finite subcovering of K , then for $U := \bigcap_{i=1}^n U_i$ we have

$$f'(U \times K) = f'\left(\bigcap_{i=1}^n U_i \times K\right) \subset f'\left(\bigcap_{i=1}^n U_i \times \bigcup_{i=1}^n W_q\right) \subset O,$$

thus U has the required property. The other direction is easily seen as well e.g. in the treatment of the exponential law in [5]. \square

The last result allows us to extend the notion of c-differentiability to the notion of **Keller differentiability** or **k-differentiability** for short, which *will* allow for iterative definitions of higher derivatives:

Definition 3.20 *Let F, G be two Fréchet spaces, $U \in F$ open. If $Q : U \rightarrow G$ is differentiable at all points $p \in U$, if $dQ(P)$ is continuous for all $p \in U$ and if the induced map $Q' : U \times F \rightarrow G$ is continuous, we call Q **k-differentiable**. The space of all k -differentiable maps from U to G is denoted by $K(U, G)$. Inductively we can define $Q^{(k+1)} : U \times F^{k+1} \rightarrow G$ by $Q^{(k+1)}(u, f_1, \dots, f_{k+1}) = \frac{d}{dt} Q^{(k)}(u + t f_{k+1}, f_1, \dots, f_k)$, for every k as this consists of continuous maps and is continuous. The space of maps $Q : U \rightarrow G$ for which $Q^{(i)}$ exists is denoted by $K^i(U, G)$.*

Definition 3.21 *Let $(F, d), (G, d')$ be two metric Fréchet spaces, $r \in (0, \infty]$. The space $B_r(F, d, G, d')$ is the space of all r -bounded linear maps between F and G , that means, all maps f for which*

$$d_{F,G}^{(r)}(f, 0) := \langle f \rangle_r := \sup_{p \in B_r(0) \setminus \{0\}} \frac{d(f(p), 0)}{d(p, 0)}$$

is finite. If no confusion can occur, we will denote this space also by $B_r(F, G)$.

Remark. Of course always $r \leq R \Rightarrow B_R(F, G) \subset B_r(F, G)$. If d_G is finite, for every $\epsilon > 0$, any given map is bounded on $F \setminus B_\epsilon(0)$ by ϵ^{-1} , thus $B_r(F, G) = B_R(F, G)$ as subsets of $CL(F, G)$ for all $r, R \in \mathbb{R} \cup \{\infty\}$, as for $A \in B_r(F, G)$ we have $\langle A \rangle_R \leq \max\{\langle A \rangle_r, r^{-1}\}$. Nevertheless, although for d_G finite they coincide as subsets of $CL(F, G)$, the spaces $B_r(F, G)$ carry different topologies. Note that $B_r(F, G)$ can be made a (locally convex, if (G, d') has arbitrarily small convex balls) complete metric vector group with metric $\delta(f, g) := \langle f - g \rangle$; it is, however, in general *not* a Fréchet space and not even a topological vector space as was pointed out in [7] where it was proven that the obvious (pointwise) scalar multiplication is not continuous in $B_\infty(F, G)$ at $t = 0$ if F or G contains a line through 0 on which the metric is bounded. Actually one can formulate an even more saddening result:

Proposition 3.22 *Let F, G be Fréchet spaces. Let d_1 be a metric on F and d_2 be a metric on G which generate the respective topologies. Then for every $r \in (0, \infty]$, if G is not Banachable and $B_r(F, d_1, G, d_2)$ is a topological vector space, it does not contain any surjective map with a bounded right inverse. If F is not Banachable and $B_r(F, d_1, G, d_2)$ is a topological vector space, it does not contain any map bounded away from zero. In the particular case of $(F, d_1) = (G, d_2)$, $B_r(F, d, F, d)$ is not a topological vector space, unless F is Banachable.*

Proof. First we show that for a non-Banachable metric Fréchet space H , for every real numbers K, r there are vectors $h \in B_r(0)$ with $l_K(h) := d(Kh)/d(h)$ arbitrarily close to one. Namely, if we could bound l_K from 1 by, say, $\epsilon > 0$, take U_0 a convex reflection-symmetric open neighborhood of 0 contained in $B_1(0)$ and (let w.r.o.g. $K > 1$) $U_n := \{h \in H | K^n \cdot h \in U_0\}$. As $U_n \subset B_{(1-\epsilon)^n}(0)$, this would be a neighborhood base for 0, and it is easy to check that the norm $\|v\| := \inf\{\lambda \in \mathbb{R} | \lambda^{-1} \cdot v \in U_0\}$ would generate the topology of H . Therefore we cannot bound l_K from 1, and there are $v_n \in B_r(0)$ with $l_K(v_n) < 1 + \frac{1}{n}$. Now consider first the case that $B_r(F, G)$ contains a surjective map f with bounded right inverse g . Then $C := \text{im}(g)$ is a Fréchet subspace of F and $f|_C$ is an isomorphism bounded both from ∞ and 0. But λf does not converge to 0 as $\lambda \rightarrow 0$, because

$$\begin{aligned}
\sup_{x \in B_r(0) \setminus \{0\}} \frac{d(\lambda f(x), 0)}{d(x, 0)} &= \sup_{x \in B_r(0) \setminus \{0\}} \left(\frac{d_2(\lambda f(x), 0)}{d_2(f(x), 0)} \cdot \frac{d_2(f(x), 0)}{d_1(x, 0)} \right) \\
&\geq \sup_{x \in B_r(0) \setminus \{0\}} \left(\frac{d_2(\lambda f(x), 0)}{d_2(f(x), 0)} \right) \cdot \inf_{x \in B_r(0) \setminus \{0\}} \left(\frac{d_2(f(x), 0)}{d_1(x, 0)} \right)
\end{aligned}$$

in which expression the first factor is 1 and the second one independent of λ . So $B_r(F, G)$ is not a topological vector space in this case.

In the other case observe

$$\begin{aligned}
\sup_{x \in B_r(0) \setminus \{0\}} \frac{d(\lambda f_2(x), 0)}{d_1(x, 0)} &= \sup_{x \in B_r(0) \setminus \{0\}} \frac{d_2(f(x), 0)}{d_1(\lambda^{-1}x, 0)} \\
&= \sup_{x \in B_r(0) \setminus \{0\}} \left(\frac{d_2(f(x), 0)}{d_1(x, 0)} \cdot \frac{d_1(x, 0)}{d_1(\lambda^{-1}x, 0)} \right) \\
&\leq \sup_{x \in B_r(0) \setminus \{0\}} \left(\frac{d_2(f(x), 0)}{d_1(x, 0)} \right) \cdot \inf_{x \in B_r(0) \setminus \{0\}} \left(\frac{d_1(x, 0)}{d_1(\lambda^{-1}x, 0)} \right)
\end{aligned}$$

and then proceed as above. For the statement about the case $F = G$ note that the identity is surjective and bounded from zero, independently of the metric used. \square

Now, as in general $B_r(F, G)$ is not a topological vector space, again we face the problem how to define higher derivatives. $B_r(F, G)$ is an algebraic subspace of $L(F, G)$. But as its topology is finer than the one induced from $L(F, G)$ it is a nontrivial question whether it is open or closed in $L(F, G)$. It turns out that in general it is never open nor closed: Take $F = \mathbb{R}^{\mathbb{N}}$ with a standard metric, and for all $j \in \mathbb{N}$ consider $f_j : F \rightarrow F$ with $f_j(v_i) = iv_i$ for all $i \leq j$ and zero on the complement C_j of the canonical embedded $\mathbb{R}^j \subset \mathbb{R}^{\mathbb{N}}$. Clearly in $L(F, F)$ the sequence converges to f with $f(v_i) = iv_i$ for all $i \in \mathbb{N}$ and zero on C . But all f_j are contained in $B_r(F, F)$ while f is not. Thus $B_r(F, F)$ is not closed in $L(F, F)$. On the other hand, 0 is contained in $B_r(F, F)$, and $F_j := f - f_j$ converges to 0 in $L(F, F)$ while all F_j are unbounded. Therefore $B(F, F)$ is not open in $L(F, F)$, either.

We recall that, following Glöckner's result, for Fréchet spaces F, G , the space $B_r(F, G)$ is not a topological vector space in general, but a complete metric vector group with absolutely convex balls (if we consider real Fréchet spaces, this means that the balls are invariant under reflections $x \mapsto -x$. In the complex case, they have to be invariant under multiplication with a complex unit). We call such a vector group a **strong vector group**. It is easy to see that if F, H are strong vector groups then $B_r(F, H)$ is again a strong vector group. Although we cannot really use $B_r(F, G)$ in order to *define* higher derivatives as this requires a topological scalar product, we can use the Keller derivatives and *reinterpret* them in the bounded sense: Define inductively strong vector groups $B_r^{(i+1)}(F, G) := B_r(F, B_r^{(i)}(F, G))$ and spaces of set-theoretic linear maps $L^{i+1}(F, G) := L(F, L^i(F, G))$. Then, for $PL(U \times F^k, G)$ denoting the space of maps which are linear in every argument except the first one, consider natural maps $T^k : PL(U \times F^k, G) \rightarrow L^k(F, G)^U$. This enables us to make the following definition:

Definition 3.23 *Let F, G be Fréchet spaces, let $U \subset F$ be open, and let $Q : U \rightarrow G$ be a map. Q is called **bounded-differentiable** or **b-differentiable with radius R** if it is differentiable at every point, if all maps $dQ(p)$, $p \in U$, are bounded within the ball around 0 of radius R and if the map $dQ : U \rightarrow B_R(F, G)$ is continuous. For $i \in \mathbb{N} \cup \{\infty\}$, we define Q to be i times b-differentiable with radius r in U if $Q \in K^i(U, G)$ we have that all $T^k(Q^{(k)}) : U \rightarrow L^k(F, G)$ take their image in $B_r^{(k)}(F, G)$ for $k \leq i$ and are continuous w.r.t. the strong vector group topology. The space of all i times b-differentiable maps with radius R we denote by $B_R^i(U, G)$.*

Of course the notion of b-differentiability is the strongest one and implies k-differentiability in very much the same way as uniform continuity implies continuity. A trivial corollary from the definition is

Theorem 3.24 *Let $A : F \rightarrow G$ be an affine map, $A(f) = Lf + g$ for all $f \in F$ where L is a linear map and $g \in G$. Then $A \in K^\infty(F, G)$ and $A'_f = L$ for all $f \in F$. Consequently, the map A is b-differentiable with radius r if and only if L is r -bounded. \square*

H.Glückner proved in [7] that the topology of $B_\infty(F, G)$ is totally disconnected. The connectivity properties of $B_r(F, G)$ for finite r is a little bit better:

Theorem 3.25 *The connected component $B_{r,0}(F, G)$ of 0 in $B_r(F, G)$ contains all maps in $B_r(F, G)$ which map $B_r(0)$ into a compact set. If $G = \Gamma(\pi)$ with a metric of the form 7 or 8, $B_{r,0}(F, G)$ equals the set of all maps in $B_r(F, G)$ which map $B_r(0)$ into a compact set.*

Proof. From [7], Prop. 2.10, we recall that $A \in B_{r,0}(F, G)$ if and only if $\lim_{t \rightarrow 0} d(tA, 0) = 0$. If $A(B_r(0))$ is precompact, all Minkowski functionals are bounded on $A(B_r(0))$, thus it is easy to see that the limit is 0. If $G = \Gamma(\pi)$, then there is a C^k norm which is unbounded on $A(B_r(0))$. Therefore the corresponding Minkowski functional is unbounded as well, so the limit is not zero. \square

Theorem 3.26 *For a vector $v \in \Gamma(\pi)$, the curve $c : t \mapsto t \cdot v$ is b-differentiable if and only if there is a real number M such that $\|v\|_k \leq M$ for all $k \in \mathbb{N}$.*

Proof. The derivative of c is $\dot{c}(t)(s\partial_t) := s \cdot v$, and one finds

$$\sup \langle sv \rangle / s = \lim_{s \rightarrow 0} \langle sv \rangle / s = d/ds|_{s=0} \langle sv \rangle.$$

The corresponding sum converges exactly if the sequence of seminorms is bounded. \square

As in [8] explained, for any Fréchet space F , the Riemannian integral along curves can be defined by its universal property that it commutes with linear functionals; it has similar properties as in finite-dimensional or Banach analysis, e.g. the fundamental theorem of calculus carries over to the Fréchet case. Likewise one can establish many of the usual theorems like the chain rule for differentials. For more on this, see [8].

H. Glückner proved a useful estimate linking the distances d_F , d_G and $d_{F,G}$. The assumptions of this theorem contains the requirement that d_G have absolutely convex balls.

Theorem 3.27 ([7], Lemma 1.11) *Let (F, d_F) and (G, d_G) be metric Fréchet spaces such that d_G has absolutely convex balls. Let $U \subset B_R(x_0) \subset F$ be a convex subset with non-empty interior and $f \in C^1(U, G)$. Then for all $x, y \in U$ we have*

$$d_G(f(x), f(y)) \leq d_F(x, y) \cdot \sup_{t \in [0,1]} \langle f'(x + t(y - x)) \rangle_R.$$

Now, an easy consequence of the established terminology which can be proved in complete analogy to the finite-dimensional case with the usual 3ϵ -arguments is the following theorem:

Theorem 3.28 *Let F, G be Fréchet spaces, let $U \subset F$ be open, let $f_n : F \rightarrow G$ be a pointwise convergent sequence of $B_r^1(U, G)$ maps with one and the same bound B and let all df_n be Lipschitz functions with one and the same Lipschitz constant B . Then the map f of pointwise limits is bounded-differentiable in U with the same bound B , and $\lim_{n \rightarrow \infty} f'_n = f'$. \square*

Remark. If all f_n are $B_r^2(U, G)$ with the same bound B , the Lipschitz condition is satisfied automatically.

Now we want to show that there are interesting examples of smooth bounded maps between Fréchet spaces:

Example 1:

Theorem 3.29 Let $\pi : E \rightarrow M$ be a vector bundle with a metric and a metric connection. For a parallel vector field X on M , the map $\nabla_X : \Gamma(\pi) \rightarrow \Gamma(\pi)$ is bounded by $2\|X\|_0$ in both standard and supremum metrics.

Proof. As $\langle \nabla^{(k)}(\nabla_X \gamma) \rangle \leq \langle \nabla^{(k+1)} \gamma \rangle$, the problem reduces to the boundedness of the shift $\sigma, (\sigma(a))_n := a_{n+1}$ in $\mathbb{R}^{\mathbb{N}}$ which is easily shown to be bounded by 2. \square

Theorem 3.30 For every unbounded metric generating the standard Fréchet topology on $\mathbb{R}^{\mathbb{N}}$ the shift is unbounded as well.

Proof. Let d be the metric in question. The sequence of open subsets $U_n := \{a \in \mathbb{R}^{\mathbb{N}} \mid a_i < 1/n \text{ for all } i \leq n\}$ is a neighborhood basis of 0. Therefore there is an $m \in \mathbb{N}$ with $U_m \in B_1^d(0)$. Choose $v \in \mathbb{R}^{\mathbb{N}}$ with $d(v, 0) > \sigma^m$. But

$$\left(\frac{1}{2m}, \frac{1}{2m}, \dots, \frac{1}{2m}, v_1, v_2, \dots\right) \in U_m \subset B_1^d(0)$$

(where the first m coordinates are meant to be $1/2m$) which is a contradiction. \square

Example 2: Let π, ψ be vector bundles over M and let $A \in \Gamma((\pi^*)^n \times \psi)$ (a polynomial with values in ψ). If A is parallel w.r.t. the canonical connection in the tensor product, then for every collection of parallel vector fields X_i^j on M the map

$$C_A : \gamma \mapsto A(\nabla_{X_1^1} \dots \nabla_{X_1^{n(1)}} \gamma, \dots, \nabla_{X_m^1} \nabla_{X_m^{n(m)}} \gamma)$$

is B_r^∞ for every radius r . This is easily seen from the facts that polynomials are bounded in every ball, that the covariant derivative is bounded as above, and that the contraction with a parallel tensor A is bounded as $\nabla_X(A(\gamma)) = A(\nabla_X \gamma)$, therefore $\nabla^n A(\gamma) = A(\nabla^n \gamma)$, and $\langle C_A \rangle \leq \|A\|_0$. One example for this construction is $\gamma \mapsto \langle \gamma, \nabla_X \gamma \rangle$ for a parallel vector field X .

Counterexample: The most striking counterexample for boundedness is the composition of maps. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a smooth map, let M be a compact manifold and let $C_f : C^\infty(M, \mathbb{R}^n) \rightarrow C^\infty(M, \mathbb{R}^m)$ be the composition with f . It is well-known that C_f is a smooth tame map, but even in the case of f having compact support it is in general wrong that C_f is b-differentiable in standard or supremum metrics. This can easily be seen by taking $f := \Phi \cdot \sin(Mx_1)$ where Φ is a compactly supported function which is identically 1 in a neighborhood of the origin.

3.3 Fréchet manifolds

Due to its better differentiability we will perform the following steps only for standard metrics although they can in principle also be done for Glöckner metrics.

Definition 3.31 A Hausdorff topological space \mathcal{M} is called a **Fréchet manifold** if \mathcal{M} has an open covering $U_\alpha, \alpha \in A$ such that for every $\alpha \in A$ there exists a homeomorphism $\Phi_\alpha : U_\alpha \rightarrow V_\alpha$ to an open subset V_α of a Fréchet space F_α such that for any $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ the map

$$\Phi_\alpha \circ \Phi_\beta|_{\Phi_\beta(U_\alpha \cap U_\beta)} : \Phi_\beta(U_\alpha \cap U_\beta) \rightarrow \Phi_\alpha(U_\alpha \cap U_\beta)$$

is a smooth map between two open sets of Fréchet spaces. If there is a Fréchet metric in each of these Fréchet spaces and if we require additionally the maps $\Phi_\alpha \circ \Phi_\beta|_{\Phi_\beta(U_\alpha \cap U_\beta)}$ to be bounded w.r.t. these metrics, we will speak of a **bounded Fréchet manifold**. If there is a uniform bound for these chart transitions we speak of a **strongly bounded Fréchet manifold**.

Definition 3.32 A **compatible metric** on a Fréchet manifold M is a metric d on M such that there is a Fréchet subatlas of M such that in each chart U , d is equivalent to the Fréchet metric d_U , that is, there are constants b_U, B_U with $b_U \cdot d_U(p, q) \leq d(p, q) \leq B_U \cdot d_U(p, q)$ for any two points in U , and if d is scalar-bounded in every chart. If there is a choice of b_U, B_U independent of U we speak of a **strong compatible metric**.

This definition implies that the metric d generates the topology of M . Moreover, an important property of d implied by the definition is that for every point of M there is a real number R_p such that Cauchy sequences in a ball $B_r(p)$ have a limit in the closure of the ball as long as $r \leq R_p$. An easy consequence of the definition of a compatible metric is the following theorem:

Theorem 3.33 If a Fréchet manifold carries a (strong) compatible metric, then it is (strongly) bounded.

To make use of the notion of Fréchet manifolds (which we want to model also the spaces of sections of bundles over non-compact manifolds), we have to make sure that the bundles we are working with are geometrically not too wild. To this purpose we use the notion of bounded geometry introduced in section 2.3.

Theorem 3.34 Let (M, g) be a Riemannian manifold and $\pi : E \rightarrow M$ a fiber bundle over (M, g) equipped with a Riemannian fiber bundle metric $\langle \cdot, \cdot \rangle$ and a fibre bundle connection D such that the bundle is of (strongly) bounded geometry. Then the space of smooth sections $\Gamma(\pi)$ can be given the structure of a bounded Fréchet manifold with a (strong) compatible metric.

Proof. First we define a metric on $\Gamma(\pi)$ whose values are bounded by 1. For two sections $\gamma, \delta \in \Gamma(\pi)$ which are homotopic to each other by a smooth homotopy $H \in \Gamma(\pi \times [0, T])$ of sections $\gamma_t = H(\cdot, t)$ of π , we define

$$H_m(p, t) := \langle \nabla^{(m)} \partial_t H(p, t), \nabla^{(m)} \partial_t H(p, t) \rangle_{(\tau_M^*)^{\otimes m} \otimes \tau(\pi)}$$

in which formula $\nabla = \nabla^{\gamma_t^* \tau_E}$ which is well-defined as all γ_t are immersions into E being sections of π . Having defined $H_m(p, t)$, we define

$$H^{(n, k)} := \sum_{m=0}^n \|\sqrt{\|H_m\|_{L^k(M_t)}\|_{L^1([0, T])}}\|$$

$$\langle H \rangle^{(k, \alpha)} := \langle \{H^{(i, k)}\}_{i \in \mathbb{N}} \rangle_\alpha$$

and finally

$$d^{(k, \alpha)}(\gamma_0, \gamma_1) := \inf_{H: \gamma_0 \rightsquigarrow \gamma_1} \langle H \rangle^{(k, \alpha)}$$

We will omit the indices (k, α) whenever this does not cause any confusion. Sometimes, if we refer to a special metric g in the fibre, we will use the notion d_g . Obviously, the function $d^{(k)}$ is **symmetric** by definition. It is **positive**, as all its defining terms are positive and as the function $\Phi : x \mapsto \frac{x}{1+x}$ maps the positive semiaxis to itself. The vanishing of $d^{(k)}$ implies that $\gamma = \delta$ as $H^{(0, k)} \geq \int \text{dist}_f(\gamma(m), \delta(m)) dm$ where dist_f denotes the fibrewise Riemannian distance. This quantity is greater than zero for $\gamma \neq \delta$. The **triangle inequality** follows by puzzling together two isometries: if two sections γ_0, γ_1 are homotopic to each other by an homotopy H_{01} then we can find another homotopy H'_{01} between them all of whose t -derivatives vanish at $t = 1$ (by concatenating H_{01} with a diffeomorphism ψ_{01} of $[0, 1]$ all of whose derivatives vanish at 1) and still with $\langle H'_{01} \rangle_p = \langle H_{01} \rangle_p$ by reparametrization-invariance of the arc-length quantities $H^{(n, k)}$. The same works for an homotopy H_{12} between γ_1 and γ_2 ; here the reparametrized homotopy H'_{12} having vanishing t -derivatives at $t = 1$. Now defining an homotopy H_{02} between γ_0 and γ_2 by the prescription $H_{02} := H'_{01}$ for $t \in [0, 1]$ and $H_{02} := H'_{12}$ for $t \in [1, 2]$, we can show by elementary calculus that $H_{02}^{(n)} = H_{01}^{(n)} + H_{12}^{(n)}$ for all n , and the convexity of $\Phi : x \mapsto \frac{x}{1+x}$ implies $\langle H_{02} \rangle \leq \langle H_{01} \rangle + \langle H_{12} \rangle$ which in turn by infimum arguments implies the triangle inequality.

The **Fréchet manifold structure** is provided by smoothly finite sections (w.r.t. d_γ) of $\gamma^*\tau_M^v$ around a section γ of π . The topology will be induced by $U \in \tilde{\Gamma}(\pi)$ open if $\kappa^{-1}(U)$ open for all charts κ of the atlas. We equip an open set of the pulled-back vertical bundle $\gamma^*\tau_E^v$ with two different metrics g_γ resp. g by pullback along γ resp. along the identification with an open set in $\Gamma(\pi)$ by the exponential map in the fibre. Then we have to prove the following proposition:

Proposition 3.35 *The metric d and the translational-invariant pullback metric $d_{g_\gamma} =: d_{(\gamma)}$ are topologically equivalent as metrics on sufficiently small $d_{(\gamma)}$ -open subsets U of $\tilde{\Gamma}_{d_{(\gamma)}}(\gamma^*\tau_M^v)$, i.e. every $d_{(\gamma)}$ -ball around a section in U contains a d -ball and vice versa.*

Proof. First we need to prove a lemma which shows in the same time that in the case of π being a vector bundle with translational-invariant metric and vector bundle connection the metric we define is the same as the metric defined in the section about Fréchet spaces:

Lemma 3.36 *If we equip $\gamma^*\tau_M^v$ with a vector bundle metric h (as e.g. the pullback metric g_γ) and a metric vector bundle connection (as e.g. the pullback connection) we have $d_{g_\gamma}(\alpha, \beta) = \langle L \rangle_h$ where $L = (1-t)\alpha + t\beta$ is the affine homotopy joining α and β .*

Proof. As by definition of a vector bundle metric every fibre is a flat vector space with a translational-invariant metric, the minimizing homotopy has to be affine as one can see by decomposing the homotopy in an affine part and one perpendicular to it: First we put w.r.o.g. $p = 0$. As l is a continuous convex function it is enough to show that the straight line segment is a *local* minimum of l , therefore we can restrict ourselves to the open space of curves $c: 0 \rightsquigarrow q$ with $\langle q, \dot{c} \rangle > 0$. Let $\epsilon > 0$ be given. Then pick $I, N \in \mathbb{N}$ such that $2^{-I}2^{-N} < \epsilon$. Consider the decomposition $\dot{c} = fq + s$ in the Hilbert space determined by the norms up to I, N where $s \perp q$ in this Hilbert space (and so in all scalar products taking part in it). As the length functional is invariant under reparametrizations, we can parametrize these curves c such that $f = 1$. Then we have $\langle c \rangle \geq \langle g \rangle - \epsilon = d(0, q) - \epsilon$. \square

Proof of the proposition, continued. It is enough to show that for all sections $\alpha \in \tilde{\Gamma}_{g_\gamma}(\gamma^*\tau_M^v)$ and for all $\delta > 0$, there is a $\rho > 0$ such that $B_\rho^{d_{(\gamma)}}(\alpha) \subset B_\delta^d(\gamma)$ and vice versa, i.e. with the roles of d and $d_{(\gamma)}$ exchanged. First we show the inequalities between the metrics: For the charts we use Riemannian normal coordinates restricted on the open subsets appearing in Theorem 2.21.

Lemma 3.37 *Let π be of bounded geometry, pick one section γ . Then there are constants κ_n with $I_{g_\gamma}^{i,k} \leq \kappa_i I_g^{i,k}$ and $I_g^{i,k} \leq \kappa_i I_{g_\gamma}^{i,k}$ for all isotopies I which satisfy $\text{dist}^h(\gamma, I) \leq \beta$ where β is the radius of the ball in Theorem 2.21.*

Proof of the lemma. First show that the parallel transport along fiber geodesics is uniformly bounded using Theorem 2.21. Then note that $I^*\tau_E^v$ and $\gamma^*\tau_E^v$ are exactly related by this parallel transport (which corresponds to radial translation in the exponential plane) and apply Theorem 2.14 and Theorem 2.22. \square

Now we have the inequalities (for L being the linear isometry in the chart linking α with β , with κ_i as above, $\bar{\kappa}_n := \max_{i \in \{1, \dots, n\}} \kappa_i$)

$$\begin{aligned}
d(\alpha, \beta) &\leq \langle L \rangle_g \\
&\leq \sum_{i=1}^n 2^{-i} \Phi(L_g^{(n)}) + 2^{-n} \text{ (rest sum)} \\
&\leq \sum_{i=1}^n 2^{-i} \Phi(\kappa_i L_{g_\gamma}^{(i)}) + 2^{-n} \text{ (monotonicity of } \Phi, \text{ Lemma 3.37)} \\
&\leq \bar{\kappa}_n \sum_{i=1}^n 2^{-i} \Phi(L_{g_\gamma}^{(i)}) + 2^{-n} \text{ (scalar - boundedness)} \\
&\leq \bar{\kappa}_n \sum_{i=1}^{\infty} 2^{-i} \Phi(L_{g_\gamma}^{(i)}) + 2^{-n} \\
&= \bar{\kappa}_n \langle L \rangle_{g_\gamma} + 2^{-n} \text{ (Lemma 3.36)} \\
&= \bar{\kappa}_n d_\gamma(\alpha, \beta) + 2^{-n}
\end{aligned}$$

Likewise, for I being an isotopy between α and β with

$$\langle I \rangle_g < 2d(\alpha, \beta) \tag{9}$$

we get the inequality

$$\begin{aligned}
d_\gamma(\alpha, \beta) &\leq \langle I \rangle_{g_\gamma} \\
&< \sum_{i=1}^n 2^{-i} \Phi(I_{g_\gamma}^{(i)}) + 2^{-n} \text{ (rest sum)} \\
&\leq \sum_{i=1}^n 2^{-i} \Phi(\kappa_n I_g^{(i)}) + 2^{-n} \text{ (monotonicity of } \Phi, \text{ Lemma 3.37)} \\
&\leq \bar{\kappa}_n \sum_{i=1}^n 2^{-i} \Phi(I_g^{(i)}) + 2^{-n} \text{ (convexity of } \Phi) \\
&\leq \bar{\kappa}_n \sum_{i=1}^{\infty} 2^{-i} \Phi(I_g^{(i)}) + 2^{-n} \\
&\leq \bar{\kappa}_n \langle I \rangle_g + 2^{-n} \text{ (Definition of } \langle I \rangle_g) \\
&< 2\bar{\kappa}_n d(\alpha, \beta) + 2^{-n} \text{ (Condition (9))}
\end{aligned}$$

Thus, the inequalities $d(\alpha, \beta) < 2\kappa_n d_\gamma(\alpha, \beta) + 2^{-n}$ and $d_\gamma(\alpha, \beta) < 2\bar{\kappa}_n d(\alpha, \beta) + 2^{-n}$ hold true for every given natural number n . Thus let a ball $B_R^d(\gamma)$ be given, then choose n with $R > 2^{-n+1}$, then elementary calculus shows that $B_{2^{-n-1}\kappa_n}^{d_\gamma}(\alpha) \subset B_R^d(\alpha)$ which proves the proposition. \square

Now the modelling vector space is **locally convex** as the topologies induced by the translational-invariant metric d_γ and the one induced by g in any chart are equivalent, as shown above.

The metric d is **locally complete** in the sense specified above because of the inequalities established above. But the Fréchet manifold itself is in general not a complete metric space (see below). \square

Note that this construction would not be possible if in the definition of the notion of Fréchet manifold the existence of a countable basis of the topology were required.

Remark. We defined the metric d for spaces $\Gamma(\pi)$ of sections of fiber bundles. What does happen if we want to deal with maps $f : M \rightarrow N$ between manifolds? Of course, we would represent them as sections of the trivial fiber bundle $p_1 : M \times N \rightarrow M$. Now let us consider two sections γ, δ of $\Gamma(\pi)$ and forget about their property being sections, only considering them as maps from M to the total space N of π . If we denote by d^t the metric in the space of sections of the associated trivial bundle $M \times N$, it is obvious that we get $d^t(\gamma, \delta) \leq d(\gamma, \delta)$ as every homotopy as a section can be seen as a homotopy as a map, and the metrics that apply coincide. The converse is wrong in general which can be seen in the example of $C^{\mathbb{S}^1, \mathbb{R}}$ by considering to sharp peaks centered at two points very close to each other but with disjoint supports : They are very close as maps but can be arbitrarily distant as sections.

Theorem 3.38 *Let $\pi : E \rightarrow M$ be a bounded bundle. If all fibers of π are complete, then the Fréchet manifold $\Gamma(\pi)$ with the metric d_g is a complete metric space.*

Proof. As d_g contains the fiber distance as an additive term, if we evaluate a d_g -Cauchy sequence γ_i at a point $p \in M$, we get a Cauchy sequence in $\pi^{-1}(p)$ which converges to a point q . By the usual estimates in uniform convergence one gets the continuity of the pointwise limit: By C^1 -convergence of the γ_i we can conclude the uniform continuity of the sequence and for a sequence $p_i \rightarrow p$ we can decompose $d_f(\gamma(p_i), \gamma(p)) \leq d_f(\gamma(p_i), \gamma_j(p_i)) + d_f(\gamma_j(p_i), \gamma_j(p)) + d_f(\gamma_j(p), \gamma(p))$ to show that the pointwise limit γ is continuous. Then consider the sections γ_i of the Cauchy sequence now in a chart around γ with the translational-invariant metric equivalent to d_g . Now for every vector field V on M , the sequence of sections $\nabla_V \gamma_i$ is Cauchy and converges pointwise. Its continuity can be proven by C^2 -convergence of γ_i . In this manner we can proceed inductively to prove the smoothness of γ . \square

Definition 3.39 *Let $(M, d), (N, d')$ be two Fréchet manifolds with compatible metric. The space $B_r(M, d, N, d')$ is the space of all r -bounded maps between M and N , that means, all maps f for which*

$$\langle f \rangle_r := \sup_{p \neq q \in M, d(p, q) < r} \frac{d(f(p), f(q))}{d(p, q)}$$

is finite. If no confusion can occur, we will denote this space also by $B(M, N)$.

This is a generalization of the same notion in the case of Fréchet spaces.

Theorem 3.40 *Let $R > 0$ be given. In the case $(M, d) = (\Gamma(\pi), d_g), (N, d') = (\Gamma(\pi), d_{g'})$ and for a Fréchet map $f : M \rightarrow N$ we have*

$$\langle f \rangle_R := \sup_{q \in M, p \in B_r(q) \setminus \{q\}} \frac{d(f(p), f(q))}{d(p, q)}.$$

for a $r < R$, i.e., we can restrict us to the case of pairs of points with any distance smaller than R .

Proof. Let $\epsilon > 0$ be given. Define $\langle f \rangle_r := \sup_{q \in M, p \neq q \in B_r(q)} \frac{d(f(p), f(q))}{d(p, q)}$. Let I be an isotopy between two points $A, B \in M$ with $\langle I \rangle < d(A, B) + \epsilon$. Then consider the function $D(t) = \langle I|_{[0, t]} \rangle$ and split the isotopy into finitely many parts $I_i : a_i \rightsquigarrow a_{i+1}$ such that $\langle I_i \rangle < r$ for all i , in particular $d(a_i, a_{i+1}) < r$. Then we have $d'(f(a_i), f(a_{i+1})) < \langle f \rangle_r d(a_i, a_{i+1})$, and the iterative application of the triangle inequality implies that

$$d'(f(A), f(B)) < \langle f \rangle_r \cdot \langle I \rangle < \langle f \rangle_r \cdot (d(A, B) + \epsilon),$$

for arbitrary $\epsilon > 0$, and the claim follows. \square

Note that in the case of M, N being Fréchet spaces with fixed Fréchet metrics and f being linear, we have $\langle f \rangle_R := \sup_{p \in B_r(0)} \frac{d(f(p), 0)}{d(p, 0)}$ for all $r \leq R$.

Now let us rephrase the inverse function theorems of Nash and Moser in the language of the metric-tame category:

Theorem 3.41 (Left inverse function Theorem between Fréchet spaces) *Let F and G be metric-tame Fréchet spaces. Let $U \subset F$ be open, $f : U \rightarrow G$ smooth metric-tame, let $f'(x)$ be injective in U with left inverse $L_0 : U \times G \rightarrow F$ which is metric-tame in U as well. Then there is a real number $r_0 > 0$ such that there is a continuous left inverse of $f|_{B(x_0, r_0)}$ (thus in particular it is injective). \square*

Theorem 3.42 (Right inverse function Theorem between Fréchet spaces) *Let F and G be metric-tame Fréchet spaces. Let $U \subset F$ be open, $f : U \rightarrow G$ smooth metric-tame with surjective differential in U and smooth metric-tame right inverse $R_0 : U \times G \rightarrow F$. Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in G and there is a smooth right inverse of $f|_{B(x_0, r_0)} \rightarrow V$ (thus in particular it is surjective). \square*

Theorem 3.43 (Full inverse function Theorem between Fréchet spaces) *Let F and G be metric-tame Fréchet spaces. Let $U \subset F$ be open, $f : U \rightarrow G$ smooth metric-tame, let $f' : U \times F \rightarrow G$ be a smooth metric-tame map which is an isomorphism at every $u \in U$. Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in G and $f|_{B(x_0, r_0)} \rightarrow V$ is a diffeomorphism. \square*

As corollaries, we get the inverse function theorems for Fréchet manifolds with compatible metrics:

Theorem 3.44 (Left inverse function theorem between metric Fréchet manifolds) *Let M resp. N be Fréchet manifolds with compatible metrics d_M resp. d_N . Let $x_0 \in U$, $U \subset M$ be open, $f : U \rightarrow N$ smooth metric-tame, let $T^1(f')$ be injective in U with left inverse $L : U \times G \rightarrow F$ smooth metric-tame as well. Then there is a real number $r_0 > 0$ such that there is a continuous left inverse of $f|_{B(x_0, r_0)}$ (thus in particular it is injective). \square*

Theorem 3.45 (Right inverse function Theorem between metric Fréchet manifolds) *Let M resp. N be Fréchet manifolds with compatible metrics d_M resp. d_N . Let $x_0 \in U$, $U \subset M$ be open, $f : U \rightarrow G$ smooth metric-tame with surjective differential in U and right inverse $R : U \times G \rightarrow F$ which is smooth metric-tame as well. Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in G and there is a smooth metric-tame right inverse of $f|_{B(x_0, r_0)} \rightarrow V$ (thus in particular it is surjective). \square*

Theorem 3.46 (Full inverse function theorem between metric Fréchet manifolds) *Let M resp. N be Fréchet manifolds with compatible metrics d_M resp. d_N . Let $U \subset M$ be an open set, $f : U \rightarrow N$ smooth metric-tame w.r.t. d_M and d_N . Let $T^1(f')$ be an isomorphism in U with an inverse $I : U \times G \rightarrow F$ which is smooth metric-tame as well. Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in N and $f|_{B(x_0, r_0)} \rightarrow V$ is a diffeomorphism. \square*

4 An inverse function theorem for bounded maps

In this section we want to state an inverse function theorem in the category of Fréchet spaces and bounded maps. We will go very closely along the lines of a nice pedagogical introduction to the same subject in *Banach* spaces written by Ralph Howard ([9]). The results are interesting due to the fact that unlike the Nash-Moser theorem invertibility is required only in one point. An interesting extension of the inverse function theorems given below was given by Helge Glöckner in [7], there only finite differentiability is required.

Theorem 4.1 (Von-Neumann series) *Let (F, d) be a metric Fréchet space of finite metric and $A \in B_R(F, d, F, d)$ with $\langle A - \mathbf{1}_F \rangle_R = r < 1, r < R$. Then A is invertible with inverse*

$$A^{-1} = \sum_{i=0}^{\infty} (\mathbf{1}_F - A)^i$$

which satisfies the estimate

$$\langle A^{-1} \rangle_R \leq \frac{1}{1 - \langle \mathbf{1}_F - A \rangle_R}.$$

Moreover for $0 < \rho < 1/2$, $\langle A^{-1} \rangle < 1$, $\langle \mathbf{1}_F - A \rangle_R, \langle \mathbf{1}_F - B \rangle_R \leq \rho$ and $\langle A - B \rangle_R < 1/2$ we have $\langle A^{-1} - B^{-1} \rangle_R \leq \frac{1}{(1-\rho)^2} \langle A - B \rangle_{2R}$.

Proof. Let $B := \sum_{k=0}^{\infty} (\mathbf{1}_F - A)^k$. Then $\langle (\mathbf{1}_F - A)^k \rangle_R \leq \langle \mathbf{1}_F - A \rangle_R^k = r^k$, so the series defining B converges by comparison with the geometric series on $B_R(0)$ and by linear continuation on all of F , and

$$\langle B \rangle_R \leq \sum_{k=0}^{\infty} \langle \mathbf{1}_F - A \rangle_R^k = \frac{1}{1 - \langle \mathbf{1}_F - A \rangle_R}.$$

Now compute

$$\begin{aligned} AB &= \sum_{k=0}^{\infty} A(\mathbf{1}_F - A)^k = \sum_{k=0}^{\infty} (\mathbf{1}_F - (\mathbf{1}_F - A))(\mathbf{1}_F - A)^k \\ &= \sum_{k=0}^{\infty} (\mathbf{1}_F - A)^k - \sum_{k=0}^{\infty} (\mathbf{1}_F - A)^{k+1} = \mathbf{1}_F. \end{aligned}$$

Likewise one can show $BA = \mathbf{1}_F$ or simply use that A and B commute.

Now if $\langle \mathbf{1}_F - A \rangle_R, \langle \mathbf{1}_F - B \rangle_R \leq \rho$, the preceding shows $\langle A^{-1} \rangle_R, \langle B^{-1} \rangle_R \leq \frac{1}{1-\rho} < 2$, and

$$\begin{aligned} \langle B^{-1} - A^{-1} \rangle_R &= \langle B^{-1}(B - A)A^{-1} \rangle_R \leq \langle B^{-1} \rangle_R \langle B - A \rangle_R \langle A^{-1} \rangle_R \\ &\leq \frac{1}{(1-\rho)^2} \langle A - B \rangle_R \end{aligned}$$

which completes the proof. \square

Theorem 4.2 *Let F and G be metric Fréchet spaces and let $A, B \in B(F, G)$ and A be invertible, $A^{-1} \in B_r(G, F)$ with $\langle A \rangle_r \leq 1$. Then if $\langle A - B \rangle_r < \frac{1}{2\langle A^{-1} \rangle_r}, 1$, then B is invertible as well and*

$$\langle B^{-1} \rangle_r \leq \frac{\langle A^{-1} \rangle_r}{1 - \langle A^{-1} \rangle_r \langle A - B \rangle_r}$$

and

$$\langle B^{-1} - A^{-1} \rangle_r \leq \frac{\langle A^{-1} \rangle_r^2 \langle B - A \rangle_r}{1 - \langle A^{-1} \rangle_r \langle A - B \rangle_r}.$$

Thus the set U of r -bounded-invertible maps r -bounded by 1 from F to G is open in $\{A \in B_r(F, G) \mid \langle A \rangle_r < 1\}$, and the map $A \mapsto A^{-1}$ is continuous on U .

Proof. As $B = A(\mathbf{1}_F - A^{-1}(A - B))$ and $\langle A^{-1}(A - B) \rangle_r \leq \langle A^{-1} \rangle_r \langle A - B \rangle_r < 1$ by assumption, Theorem 4.1 shows that $\mathbf{1}_F - A^{-1}(A - B)$ is invertible and that

$$\langle (\mathbf{1}_F - A^{-1}(A - B))^{-1} \rangle_r \leq \frac{1}{1 - \langle A^{-1}(A - B) \rangle_r} \leq \frac{1}{1 - \langle A^{-1} \rangle_r \langle A - B \rangle_r}.$$

As we assumed A to be invertible, $B = A(\mathbf{1}_F - A^{-1}(A - B))$ is invertible as well with $B^{-1} = (\mathbf{1}_F - A^{-1}(A - B))^{-1}A^{-1}$ and

$$\langle B^{-1} \rangle_r \leq \langle (\mathbf{1}_F - A^{-1}(A - B))^{-1} \rangle_r \langle A^{-1} \rangle_r \leq \frac{\langle A^{-1} \rangle_r}{1 - \langle A^{-1} \rangle_r \langle A - B \rangle_r}.$$

Finally, $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$ gives

$$\langle B^{-1} - A^{-1} \rangle_r \leq \langle A^{-1} \rangle_r \langle A - B \rangle_r \langle B^{-1} \rangle_r \leq \frac{\langle A^{-1} \rangle_r^2 \langle B - A \rangle_r}{1 - \langle A^{-1} \rangle_r \langle A - B \rangle_r}$$

which completes the proof. \square

Theorem 4.3 *Let F and G be Fréchet spaces, let $U \subset F$ be open. Let $h \in B_r^2(U, G)$, let $dh(p)$ be bounded-invertible, let $I : \rightarrow B(G, F)$ be defined by $I(X) = X^{-1}$. Then $J := I \circ dh : V \rightarrow B(G, F)$ is b-differentiable on a possibly smaller open neighborhood V of p , and for $q \in V$, the derivative $dJ : V \rightarrow B_r(F, B_r(G, F))$ is given by*

$$dJ(v)f = -(dh(v))^{-1} \circ dh'(v, f) \circ (dh(v))^{-1}.$$

Proof. As the set of bounded-invertible maps W is an open set in $B(F, G)$ according to Theorem 4.2, and as $dh : U \rightarrow B(F, G)$ is continuous, h is bounded-invertible in an open neighborhood of p . Let $L : B(F, G) \rightarrow B(G, F)$ be the linear map given by $Lf := -A^{-1}fA^{-1}$ which is continuous on W (Theorem 4.2 again). Now we will show that this expression gives the derivative of the inversion in its domain of definition in $CL(F, G)$. So for $X \in W$,

$$\begin{aligned} I(X) - I(A) - L(X - A) &= X^{-1} - A^{-1} + A^{-1}(X - A)A^{-1} \\ &= X^{-1}(A - X)A^{-1} + A^{-1}(X - A)A^{-1} \\ &= (-X^{-1} + A^{-1})(X - A)A^{-1} \\ &= X^{-1}(X - A)A^{-1}(X - A)A^{-1}, \end{aligned}$$

thus for $X = A + tB$ we get

$$\begin{aligned} (I(X) - I(A) - L(X - A))/t &= ((A + tB)^{-1}tBA^{-1}tBA^{-1})/t \\ &= t(A + tB)^{-1}BA^{-1}BA^{-1} \end{aligned}$$

As I is continuous according to Theorem 4.2, we have $\lim_{t \rightarrow 0}(A + tB)^{-1} = A^{-1}$ and

$$\lim_{t \rightarrow 0} \frac{I(X) - I(A) - L(X - A)}{t} = \lim_{t \rightarrow 0} t(A + tB)^{-1}BA^{-1}BA^{-1} = 0,$$

thus for dJ we get pointwise the expression in the theorem (note that we could not have done the calculation directly in $B_r(F, G)$ as it uses the topological vector space structure). This expression is bounded in f as $h \in B_r^2(F, G)$, and continuous in v by the same reason. This completes the proof. \square

Theorem 4.4 (Banach's fix point Theorem) *Let (X, d) a complete metric space and $f : X \rightarrow X$ a contraction with contraction factor $\rho < 1$. Then f has a unique fix point x_f in X . It is the limit of the recursive sequence $x_0 \in X$ arbitrary, $x_{n+1} = f(x_n)$. The distance to the solution decreases like*

$$d(x_n, x_f) \leq \frac{\rho^n}{1 - \rho} d(x_0, x_1).$$

\square

Theorem 4.5 (Left inverse function Theorem between Fréchet spaces) *Let F and G be metric Fréchet spaces with a metric of the form (8). Let $x_0 \in U$, $U \subset F$ be open, $f : U \rightarrow G$ b-differentiable with radius R , let $f'(x_0)$ be injective with R -bounded left inverse $L_0 \in B_R(G, F)$. Then there is a real number $r_0 > 0$ such that there is a continuous left inverse of $f|_{B(x_0, r_0)}$ (thus in particular it is injective).*

Proof. This is one of the rare occasions where we use local convexity: We choose a convex subset $U_c \subset U$ containing x_0 . We consider the map $\xi : U \rightarrow B_R(F, F)$, $\xi : x \mapsto L_0 \circ f'(x) - \mathbf{1}_F$ which is bounded-differentiable on U and vanishes at x_0 . Fix ρ with $0 < \rho < 1$, then by continuity of df we can find a convex $U' \subset U_c$ containing x_0 such that for $x \in U'$

$$\langle \xi(x) \rangle_R = \langle L_0 f'(x) - \mathbf{1}_F \rangle_R \leq \rho. \quad (10)$$

thus in U' , applying Theorem 3.27 we get

$$d(L_0 f(x_2) - L_0 f(x_1), x_2 - x_1) \leq \rho d(x_1, x_2). \quad (11)$$

We compute for $x_1, x_2 \in B_r(x_0)$, $r < R$,

$$\begin{aligned} d_F(x_1, x_2) &= d_F(x_2 - x_1, 0) \\ &\leq d_F(x_2 - x_1, L_0 f(x_2) - L_0 f(x_1)) + d_F(L_0 f(x_2) - L_0 f(x_1), 0) \\ &\leq \rho d_F(x_2 - x_1, 0) + \langle L_0 \rangle_R d_G(f(x_2) - f(x_1), 0), \end{aligned}$$

therefore

$$\frac{(1-\rho)}{\langle L_0 \rangle_R} d_F(x_1, x_2) \leq d_G(f(x_1), f(x_2)) \quad (12)$$

Therefore $f|_{B(x_0, r_0)}$ is injective and has an set-theoretic left inverse $\phi := f|_{B(x_0, r_0)}^{-1}$ which takes its image in U' and for which holds $d_F(\phi(y_1), \phi(y_2)) \leq \frac{\langle L_0 \rangle_R}{1-\rho} d_G(y_1, y_2)$, therefore both $f|_U$ and its left inverse are bounded, and ϕ is continuous. \square

Theorem 4.6 (Right inverse function Theorem between Fréchet spaces) *Let F and G be metric Fréchet spaces with a metric of the form (8). Let $x_0 \in U$, $U \subset F$ be open, $f : U \rightarrow G$ b -smooth of radius r with surjective differential at x_0 and bounded right inverse $R_0 : U \in B(G, F)$ at x_0 . Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in G and there is a b -smooth (of radius r) right inverse of $f|_{B(x_0, r_0)} \rightarrow V$ (thus in particular it is surjective).*

Proof. First, to simplify the notation, by concatenating with translations in F and G , let w.r.o.g. $x_0 = 0$, $f(x_0) = 0$. Rescale the metric d_G such that $\langle f'(0) \rangle_r < 1$. Note that the image $im(R_0)$ of R_0 is a Fréchet subspace of F as $f'(0)$ is bounded and therefore R_0 is bounded from 0; thus Cauchy sequences in $im(R_0)$ are R_0 -images of Cauchy sequences in G . As in the previous proof, we use local convexity: We choose a convex subset $U_c \subset G$ containing $f(0) = 0$. We consider the map $\xi : U_c \rightarrow B(G, G)$, $\xi : x \mapsto (f'(x)R_0 - \mathbf{1}_G)$ which is bounded-differentiable on U and vanishes at 0. Fix ρ with $0 < \rho < 1$, then by continuity of df we can find a $U' \subset U_c \cap B_r(x_0)$ containing 0 with

$$\langle \xi|_{U'} \rangle_r = \langle f'|_{U'} R_0 - \mathbf{1}_G \rangle_r \leq \rho. \quad (13)$$

Likewise in U' , as $f \circ R_0$ fixes 0 and as $(f \circ R_0)' = f' \circ R_0$, we get

$$d((\mathbf{1} - f \circ R_0)y_2, (\mathbf{1} - f \circ R_0)y_1) = d(f \circ R_0(y_2) - f \circ R_0(y_1), y_2 - y_1) \leq \rho d(y_1, y_2) \quad (14)$$

for $y_1, y_2 \in U'$, again by applying Theorem 3.27. Now we choose a ball radius r_0 such that $\overline{B}(0, r_0) \subset f^{-1}(U')$. Then for $y \in G$ we define $\Phi_y : F \supset \overline{B}(0, r_0) \rightarrow F$ by

$$\Phi_y(x) = x - (R_0(f(x) - y)).$$

Note that, up to a possible factor $c > 0$ in front of the last bracket, this is exactly the discretization of the real family of operators used in the proof of the Nash-Moser Theorem in [8], p. 180. This expression is b -differentiable, and, as R_0 is injective, we have $\Phi_y(x) = x \Leftrightarrow f(x) = y$.

The theorem 3.27 ensures that for $x_1 = R_0u_1, x_2 = R_0u_2 \in \overline{B}(0, r_0) \cap \text{im}(R_0) \subset f^{-1}(U')$ we have $d_F(\Phi_y(x_1), \Phi_y(x_2)) \leq \rho d_F(x_1, x_2)$ (we use that by surjectivity of $f'(0)$ we know $u_1 = f'(0)x_1, u_2 = f'(0)x_2$):

$$\begin{aligned} d(\Phi_y(x_1), \Phi_y(x_2)) &= d(x_1 - R_0f(x_1), x_2 - R_0f(x_2)) \\ &\leq \langle R_0 \rangle_\infty d(u_1 - f(R_0u_1), u_2 - f(R_0u_2)) \\ &= \langle R_0 \rangle_\infty d((\mathbf{1} - f \circ R_0)u_1, (\mathbf{1} - f \circ R_0)u_2) \\ &\leq \langle R_0 \rangle_\infty \langle \mathbf{1} - f \circ R_0 \rangle_r d(u_1, u_2) \\ &= \langle R_0 \rangle_\infty \langle \mathbf{1} - f \circ R_0 \rangle_r d(f'(0)x_1, f'(0)x_2) \\ &\leq \langle R_0 \rangle_\infty \langle \mathbf{1} - f \circ R_0 \rangle_r \langle f'(0) \rangle_r d(x_1, x_2) \end{aligned}$$

(note that for the third line we need $d(u_1, u_2) < r$ which is guaranteed by $d(u_1, u_2) < \langle f'(0) \rangle_{r_0} \cdot r_0, r_0 < r$ and $\langle f'(0) \rangle_r < 1$). Thus for appropriate chosen ρ, Φ_y is a contraction for every $y \in G$. We will show now that it also leaves the ball $\overline{B}(0, r_0) \cap \text{im}(R_0)$ invariant for appropriately chosen y . First of all, by means of linearity of R_0 one easily checks that it leaves the subspace $\text{im}(R_0)$ invariant. Now let $\beta := \frac{1-\rho}{\langle R_0 \rangle}$ and $r_1 := \beta r_0 = \frac{1-\rho}{\langle R_0 \rangle} r_0$, then for $R_0(u) = x \in \overline{B}(x_0, r_0) \cap \text{im}(R_0)$ and $y \in \overline{B}(f(x_0), r_1)$ we have

$$d_F(0, \Phi_y(x)) = d(R_0u - R_0f(R_0u), R_0y) \leq \langle R_0 \rangle_\infty (d(u - f(R_0u), 0) + d(y, 0))$$

Therefore if $y \in \overline{B}(f(x_0), r_1)$, then $\Phi_y : \overline{B}(x_0, r_0) \rightarrow \overline{B}(x_0, r_0)$, and we can apply Banach's fix point Theorem 4.4 showing that Φ_y has a unique fixed point in $\overline{B}(x_0, r_0)$, or, equivalently, that for every $y \in \overline{B}(f(x_0), r_1)$ there is a unique $x =: \phi(y) \in \overline{B}(x_0, r_0)$ with $f(x) = y$. Therefore $\overline{B}(f(x_0), r_1) \subset f(\overline{B}(x_0, r_0))$.

Finally we show that at each $b \in V$ the map ϕ is differentiable at b with

$$\phi'(b) = f'(\phi(b))^{-1}$$

which implies inductively the smoothness of ϕ' on V as a composition of the maps ϕ, f' and the inversion of bounded-invertible linear maps (cf. Theorem 4.2) by help of the chain rule.

Differentiability of ϕ follows from $\phi(y) := \lim_{k \rightarrow 0} \phi_y^k(x)$ for any $x \in \text{im}(R_0)$ and from Theorem 3.28: To this aim, we observe

$$\frac{d}{dt} \phi_{y+tv}^k(x) = \sum_{i+j=k-1} d\phi_y^i \left(\frac{d}{dt} \phi_{y+tv}^j(\phi_y^j(x)) \right)$$

and compute $\frac{d}{dt} \phi_{y+tv}(x) = R_0v$ (independent of x) and $d\phi_y(x)R_0v = R_0v - R_0f'(R_0v) = R_0(\mathbf{1} - f' \circ R_0)(v)$ use again the contraction property of ϕ_y and the estimate of $\mathbf{1} - f' \circ R_0$ to prove the local uniform boundedness of this expression. It is easy to see that the family $(\Phi_y^k)'$ is uniformly Lipschitz in y . This enables us to use Theorem 3.28.

Now, knowing differentiability of ϕ the equation $\phi'(b) = f'(\phi(b))^{-1}$ is an easy consequence from $f \circ R = \mathbf{1}_G$ (although it is a nice exercise to show it using the limit above and the Neumann series: As $\Phi_y^k(x) \in \text{im}(R_0)$ for all k and all $x \in \text{im}(R_0)$, and as $\text{im}(R_0)$ is closed, we know that ϕ takes its image in $\text{im}(R_0)$ as well. Therefore we can consider $\phi_R := \mathbb{R}_0^{-1} \circ \Phi : G \supset B(0, r_1) \rightarrow G$. As R_0 is linear and injective, $\phi'_R = R_0^{-1} \circ \phi'$ which is sufficiently near the identity etc.). \square

A corollary of the two preceding theorems is (as in this case right-inverse and left-inverse coincide)

Theorem 4.7 (Full inverse function Theorem between Fréchet spaces) *Let F and G be metric Fréchet spaces with the metric being a countable sum of continuous functions of seminorms as given by the equation (7). Let $x_0 \in U, U \subset F$ be open, $f : U \rightarrow G$ b -smooth, let $f'(x_0)$ be an isomorphism bounded from 0. Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in G and $f|_{B(x_0, r_0)} \rightarrow V$ is a diffeomorphism. \square*

As corollaries, we get the inverse function theorems for Fréchet manifolds with compatible metrics:

Theorem 4.8 (Left inverse function theorem between metric Fréchet manifolds) *Let M resp. N be Fréchet manifolds with compatible metrics d_M resp. d_N modelled on metric Fréchet spaces F, G with the metric being a countable sum of continuous functions of seminorms as given by the equation (7). Let $x_0 \in U$, $U \subset M$ be open, $f : U \rightarrow N$ bounded-differentiable with $f'' : U \times F \rightarrow B(F, G)$ bounded as well in U , let $f'(x_0)$ be injective with bounded left inverse $L_0 \in B(G, F)$. Then there is a real number $r_0 > 0$ such that there is a continuous left inverse of $f|_{B(x_0, r_0)}$ (thus in particular it is injective). \square*

Theorem 4.9 (Right inverse function Theorem between metric Fréchet manifolds) *Let M resp. N be Fréchet manifolds with compatible metrics d_M resp. d_N modelled on metric Fréchet spaces F, G with the metric being a countable sum of continuous functions of seminorms as given by the equation (7). Let $x_0 \in U$, $U \subset M$ be open, $f : U \rightarrow G$ bounded-differentiable with surjective differential at x_0 and bounded right inverse $R_0 : U \in B(G, F)$ at x_0 and with $f'' : U \times F \rightarrow B(F, G)$ bounded as well in U . Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in G and there is a smooth right inverse of $f|_{B(x_0, r_0)} \rightarrow V$ (thus in particular it is surjective). \square*

Theorem 4.10 (Full inverse function theorem between metric Fréchet manifolds) *Let M resp. N be Fréchet manifolds with compatible metrics d_M resp. d_N modelled on metric Fréchet spaces F, G with a metric of the form (8). Let $U \subset M$ be an open set contained in a chart, $f : U \rightarrow N$ bounded-differentiable and $f'' : U \times F \rightarrow B(F, G)$ bounded as well, everything w.r.t. d_M and d_N . Let $f(x_0)$ be an isomorphism bounded from 0. Then there is a real number $r_0 > 0$ such that $V := f(B(x_0, r_0))$ is open in N and $f|_{B(x_0, r_0)} \rightarrow V$ is a diffeomorphism. \square*

5 Fréchet manifolds as generalized length spaces

Now let us equip the Fréchet spaces and manifolds with structures carrying still a little bit more information trying to give them a notion of curve length. We recall the notion of **length spaces**, adding two generalizations:

Definition 5.1 *A length structure on a topological space X is a subset C of $\bigcup_{(a,b \in \mathbb{R}^2)} C^0([a, b], X)$ closed under concatenation of paths, restrictions to subsets and linear reparametrizations, and a function $L : C \rightarrow \mathbb{R} \cup \{\infty\}$ which is additive under concatenation of curves, invariant under linear homeomorphisms and with the additional properties that for a curve $c : [a, b] \rightarrow X$ with $L(c) < \infty$ the function $l : [a, b] \rightarrow \mathbb{R}^+, t \mapsto L(c|_{[a, t]})$ is continuous, and that for every point $x \in X$ and every open neighborhood U of x there is an $\epsilon > 0$ such the length of any curve beginning at x and ending in the complement of U is greater than ϵ . A **generalized length structure of first kind** is a subset and a function satisfying all of the above, except that the function does not have to be additive under concatenation of curves. A **generalized length structure of second kind** is a subset and a function satisfying all of the above, except that the function does not have to be invariant under reparametrizations of curves. A **length space** is a metric space such that the distance between two points p, q coincides with the infimum of length of curves joining p and q .*

If X is a metric space, there is always the (in the category of metric spaces and isometries) natural choice of a length structure $C_0 := \bigcup_{a,b \in \mathbb{R}^2} C^0([a, b], X)$, $L_0 := \sup_{P \in \mathcal{P}} \sum_{i=1}^n d(c(t_{i+1}), c(t_i))$. If a curve $k : [a, b] \rightarrow X$ is Lipschitz, i.e. if $lip_c := \sup_{s,t \in [a,b], s \neq t} d(k(s), k(t))/|s-t| < \infty$, then $L_0(k) = \int_a^b lip_t dt$, where $lip_t := \lim_{\epsilon \rightarrow 0} lip(c|_{[t-\epsilon, t+\epsilon]})$ (this limit exists as its argument is monotonely decreasing in ϵ and bounded by 0). The definition of L_0 , due to Gromov, is an extension of the notion of curve length for piecewise differentiable curves in Banach manifolds. However, the following corollary from 3.26 shows that this is not the appropriate notion of length as we would have too few rectifiable curves, i.e. curves of finite length, as these would not even include all straight lines:

Corollary 5.2 *Let $g : t \mapsto tv$ be a straight line in $\Gamma(\pi)$ with a metric of the form 7 or 8, then $L_0(g) < \infty \Leftrightarrow \exists M \in \mathbb{R} : \|v\|_i \leq M\forall i$.*

So we try to construct at least *generalized* length structures for interesting spaces of curves. To this purpose we first prove a theorem about arc-length parametrization of differentiable curves:

Theorem 5.3 *Given a C^1 curve $c : [a, b] \rightarrow F$ in a Fréchet space with $\dot{c}(t) \neq 0$, we can always find a reparametrization $\tilde{c} : [0, T] \rightarrow X$ with $d_F(\dot{c}(t), 0) = 1$.*

Proof. This follows from continuity of the first Minkowski functional and the multiplicative inversion in \mathbb{R}^+ . \square

We define the **metric length** l of a continuous curve $c : \mathbb{I} \rightarrow F$ by

$$l(c) := \int_{\mathbb{I}} \sum \alpha_n \Phi(\sup_{r \in \mathbb{R} \setminus \{0\}} \|\dot{c}(t)(r\partial_t)\|_n^M) dt$$

where the $\|\cdot\|_k^M$ are the Minkowski functionals.

We define the **smooth length** L of a smooth curve by the canonical embedding $H : C^\infty(\mathbb{I}, \Gamma(\pi)) \rightarrow \Gamma(\mathbb{I} \times \pi)$ and $L(H) := \langle H \rangle$ as in the section of Fréchet manifolds.

Obviously, the smooth length gives rise to the Fréchet metric already used, but its definition needs more than the metric alone, but the whole series of seminorms. The metric length, in turn, is defined for a wider class of curves, but it induces a new metric. It seems most appropriate to define lengths of smooth curves *first*, by L , and then define the metric as infimum of curve length. We want to compare l and L which differ basically by the order of Φ and the integral along the curve:

Theorem 5.4 *Let M be a metric Fréchet manifold. $(C^\infty(\mathbb{I}, M), L)$ is a generalized length structure of first kind, satisfying only an inequality $L(c_1 \circ c_2) \leq L(c_1) + L(c_2)$. $(C^0(\mathbb{I}, M), l)$ is a generalized length structure of second kind. We have the inequality $l(c) \leq L(c)$.*

Proof. We only show $\langle I \rangle \geq l(I)$. It is enough to show the inequality in

$$\sum 2^{-n} \Phi(a_n \int_0^1 f(t) dt) \geq \int_0^1 \sum 2^{-n} \Phi(a_n f(t)) dt = \sum 2^{-n} \int_0^1 \Phi(a_n f(t)) dt,$$

as the right hand side is $\geq \int [\dots] dt$ by the inequality after the definition of length and scalar-boundedness. So if we absorb a_n into f , it is enough to show that for an arbitrary positive function S , we have

$$\Phi\left(\int_0^1 f(t) dt\right) \geq \int_0^1 \Phi(f(t)) dt. \tag{15}$$

But this is clear from the concavity of Φ and the fact that step functions are dense in $L^1(\mathbb{I})$: Let T be a step function on the unit interval with values T_1, \dots, T_n . The integral of T is a convex combination of T_1, \dots, T_n , and a concave function maps convex sets into convex sets, explicitly: $\Phi(tx + (1-t)y) \geq t\Phi(x) + (1-t)\Phi(y)$, thus by induction $\Phi(\sum t_i x_i) \geq \sum t_i \Phi(x_i)$ if $\sum t_i = 1$. Therefore Equation (15) holds, and we are done. \square

Some important questions remain open: Is there a metric such that Gromov's curve length is finite for all smooth curves? Is there a proper length structure (no generalized one) only defined by means of the metric for which all smooth curves are rectifiable? And if yes, is there one inducing the metric it started with? This shall be object of further research.

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