Codazzi spinors and globally hyperbolic manifolds with special holonomy

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Abstract

In this paper we describe the structure of Riemannian manifolds with a special kind of Codazzi spinors. We use them to construct globally hyperbolic Lorentzian manifolds with complete Cauchy surfaces, for any weakly irreducible holonomy representation with parallel spinors, t.m. with a holonomy group $G \ltimes \mathbb{R}^{n-2} \subset SO(1, n-1)$, where $G \subset SO(n-2)$ is trivial or a product of groups $SU(k)$, $Sp(l)$, G_2 or $Spin(7)$.

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Contents

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1 Introduction

The connected holonomy groups of Riemannian manifolds are well understood and there are a lot of results and methods for construction of Riemannian metrics with special holonomy (cf. [15]). Contrary to that, the classification of holonomy groups for indefinite metrics is a longtime and widely open problem, since the existence of indecomposable but non-irreducible holonomy representations makes the classification difficult. Recently, the classification of the connected holonomy groups of Lorentzian manifolds was achieved. L. Berard-Bergery and A. Ikemakhen described the structure of weakly-irreducible, non-irreducible subgroups of the Lorentz group (cf. [7] or [11]). Th. Leistner ([18], [19]) was able to classify all Lorentzian Berger algebras, which is the essential part in the classification of connected Lorentzian holonomy groups, and he realized part of them as holonomy algebras. A. Galaev ([12]) gave local analytic Lorentzian metrics for all of these Berger algebras, including that of the still missing coupled types, thereby completing the classification of connected Lorentzian holonomy groups. The next task in this line is the construction of global geometric models with special Lorentzian holonomy.

There is a class of manifolds that is very suitable for purposes of field theories, mathematical physics etc: the class of globally hyperbolic manifolds, which can be seen as a Lorentzian analog to complete manifolds in Riemannian geometry. The construction of analytic manifolds in [12] of course yields immediately the existence of globally hyperbolic metrics for these holonomies as the holonomy of an analytic manifold equals its local holonomy, and in each analytic Lorentzian manifold every point has a globally hyperbolic neighborhood. But we can sharpen the requirements a bit and try to construct *globally hyperbolic manifolds with complete Cauchy surface* which property clearly is not shared by the examples constructed by the method described above. A class of globally hyperbolic manifolds with even stronger completeness conditions (which imply e.g. strong statements about the long-time existence of Lorentzian minimal surfaces, cf. [20]) is the one of bbc manifolds which will be defined in section 2. Therefore we ask for bbc manifolds with special holonomy. In this article we give a construction method which yields such manifolds using an idea of Ch. Bär, P. Gauduchon and A. Moroianu ([1]), who constructed parallel spinors on generalized cylinders out of Codazzi spinors. We explore this method for the Lorentzian situation in detail, describe the structure of all Riemannian manifolds with imaginary Codazzi spinors to an invertible Codazzi tensor as well as the causal and holonomy properties of the Lorentzian cylinder defined by the Codazzi tensor (which will be a bbc manifold).

In section 2 we start with basic properties of Lorentzian cylinders by which we mean a Lorentzian manifold $(a, b) \times M$ with a metric g of the form $g = -dt^2 + g_t$, where g_t is a smooth family of Riemannian metrics on M. In 2.1 we give a criterion for global hyperbolicity of a Lorentzian cylinder. In 2.2 we describe basics of Lorentzian spin geometry in order to fix our conventions and notations for the rest of the paper. In 2.3 we consider special Lorentzian cylinders that are constructed out of Codazzi spinors on the Riemannian base (M, g_0). A spinor field φ on (M, g_0) is called *imaginary Codazzi spinor* if

$$\nabla^{g_0}_X \varphi = iA(X) \bullet \varphi$$

for all vector fields X, where A is a Codazzi tensor on (M, g_0), which is then uniquely determined by the spinor φ. As in [1] we prove, that φ induces a parallel spinor $\widetilde{\varphi}$ on the Lorentzian cylinder ($C := (a,b) \times M\,,\ g := -dt^2 + (1-2tA)^* g_0$) and analyze the causal type of the associated Dirac current on C. In section 3 we draw our attention to the case of invertible Codazzi tensors. In 3.1 we prove that an imaginary Codazzi spinor with invertible Codazzi tensor A to the metric g_0 corresponds to an imaginary Killing spinor for the metric $A^* g_0$. Applying the classification result for manifolds with imaginary Killing spinors we obtain the structure result for manifolds admitting imaginary Codazzi spinors with invertible Codazzi tensor A. Any such complete manifold is isometric to

$$M = \mathbb{R} \times F\,,\ g_0 = (A^{-1})^*(ds^2 + e^{-4s} g_F) \qquad (*)$$

where (F, g_F) is a complete Riemannian manifold with parallel spinors and A^{-1} is a Codazzi tensor on the warped product $\mathbb{R} \times_{e^{-2s}} F$ (Theorem 1). In 3.2 we analyze the existence of Codazzi tensors on warped products and reduce it to the question of existence of Codazzi tensors on (F, g_F). This leads to a construction method for Riemannian manifolds with imaginary Codazzi spinors. Any $(n-1)$-dimensional Riemannian manifold (F, g_F) with parallel spinors and Codazzi tensor T which has eigenvalues uniormally bounded from below gives rise to a Codazzi tensor H on the warped product $(M = \mathbb{R} \times F, g_{wp} = ds^2 + e^{-4s} g_F)$ with eigenvalues bounded away from zero. Hence on $(M, g_0 = H^* g_{wp})$ there are Codazzi spinors and on the cylinder (C, g_C) with

$$\begin{aligned} C &= C[F;H] = (a,b) \times M = (a,b) \times \mathbb{R} \times F \\ g_C &= -dt^2 + g_t = -dt^2 + (H - 2t\mathbf{1})^* g_{wp} \end{aligned}$$

parallel spinors by the construction explained above. In 3.3 we study the causal type and the holonomy of this cylinder. In Theorem 2 we prove that $C[F;H]$ is globally hyperbolic if (F, g_F) is complete. Furthermore, we show that $C[F;H]$ is flat if and only if (F, g_F) is flat. Note, that the holonomy group of a simply connected Lorentzian manifold $\widetilde{C}$ acts irreducible if and only if it is isomorphic to $SO_0(T_x\widetilde{C})$. Since we have a parallel spinor on the cylinder $C[F;H]$ which defines a nontrivial parallel vector field by its Dirac current, $C[F;H]$ can not be irreducible. Hence the cylinder $C[F;H]$ is either weakly irreducible, meaning that the holonomy representation has no *non-degenerate* invariant subspace (but a degenerate one) or $C[F;H]$ is decomposable t.m. it is (locally) the product of a Lorentzian and a Riemannian manifold. In fact, $C[F;H]$ is a Brinkman space, t.m. it admits a nowhere vanishing parallel light-like vector field. In Theorem 3 we prove that $C[F;H]$ is decomposable if (F, g_F) contains a flat factor. If (F, g_F) is (locally) the Riemannian product of irreducible (non-flat) manifolds, then $C[F;H]$ is weakly irreducible and the connected component of the holonomy group of $C[F;H]$ is isomorphic to

$$Hol^0(F, g_F) \ltimes \mathbb{R}^{n-1}.$$

This gives a construction method for globally hyperbolic (and even bbc) manifolds with complete Cauch hypersurfaces for every weakly-irreducible, non-irreducible Lorentzian holonomy representation with a fixed spinor. In section 4 we finish the paper by studying examples of Riemannian manifolds with parallel spinors and Codazzi tensors which are the building blocks of our construction.

2 Lorentzian cylinders

2.1 Definition and causal properties

Definition: A *Lorentzian cylinder* is a product manifold $C = (a, b) \times M$ with a Lorentzian metric of the form $g = -dt^2 \oplus g_t$ where g_t is a smooth family of Riemannian metrics on M parametrized over the interval (a, b), where $-\infty \leq a < 0 < b \leq +\infty$.

We want to link this notion to the notion of bbc manifolds as described in [20]:

Definition: A Lorentzian manifold (C, g) is said to be *bbc* iff

$$(C, g) \cong (\mathbb{R} \times M, g(p) = -f(p)dt^2 \oplus g_t),$$

where f is a smooth positive function on C which is *bounded* on every $\{t\} \times M$, and g_t is a family of *complete* Riemannian metrics on M, and with the additional property that the eigenvalues of $\dot{g}_t \circ g_t^{-1}$ are uniformally *bounded* for all t.

Examples: If g_t is a compact perturbation (e.g. described by pullback by a symmetric endomorphism which is the identity outside of a compactum on every $\{t\} \times M$) of a fixed complete Riemannian metric for all values of t, then for any bounded f, $(\mathbb{R} \times M, -f dt^2 \oplus g_t)$ is bbc. On the contrary, any open proper subset of the Minkowski space is not bbc as it does not contain any complete spacelike hypersurface.

Proposition 1 *([20]) The $t =$ constant hypersurfaces of a bbc manifold are Cauchy hypersurfaces. In particular, bbc manifolds are globally hyperbolic.*

We start with the causality properties of a cylinder. For that, let A_t denote the connecting endomorphism between g_t and g_0:

$$g_t = A_t^* g_0 = g_0(A_t \cdot, A_t \cdot).$$

Proposition 2 *Let $(C, g) \cong ((a, b) \times M, g = -dt^2 \oplus g_t)$ be a Lorentzian cylinder. Then (C, g) is strongly causal. If in addition the metric g_0 is complete and the eigenvalues of $(A_t)^{-1}$ are uniformally bounded on M for every $t \in (a, b)$, then (C, g) is globally hyperbolic. If in addition to both previous assumptions the eigenvalues of $\dot{A}_t \circ A_t^{-1}$ are uniformally bounded on every $\{t\} \times M$, then (C, g) is bbc.*

Proof. The function $f(t, x) = t$ is obviously a time function on (C,g), since it is strictly increasing along any future directed causal curve. Therefore, the Lorentzian cylinder is stably causal, which implies strong causality (cf. [6], chapt. 3).
Now assume additionally that the metric g_0 is complete and that for every $t \in (a, b)$ there exists a constant $c_t \in \mathbb{R}^+$ such that all eigenvalues of A_t^{-1} are uniformally bounded by $c_t < +\infty$. We want to prove global hyperbolicity of C. Following [6], p.65, knowing that we have strong causality for C, we have to show that the intersection $N := J^+(y_0) \cap J^-(y_1)$ of the causal future $J^+(y_0)$ and the causal past $J^-(y_1)$ is compact for all points $y_0 = (T_0, q_0)$ and $y_1 = (T_1, q_1)$ of C. For that, let $\{x_n\}$ be a sequence of points in N. Then $x_n = (t_n, p_n)$, where $t_n \in [T_0, T_1]$. Hence $\{t_n\}$ has a convergent subsequence. So, we may assume that $\{t_n\}$ converges to $t^* \in [T_0, T_1]$. Now, let $\gamma_n(s) =$

$(t(s), \delta_n(s)) : [0,1] \to N$ be a future directed causal curve in N from y_0 to x_n. Then by the assumption on A_t

$$c\|\delta_n'(s)\|_{g_0} < \|\delta_n'(s)\|_{g_{t(s)}} \leq t'(s)$$

for a constant $c > 0$. It follows that the length of δ_n with respect to the metric g_0 is bounded by $R = \frac{1}{c}(T_1 - T_0)$. Hence, all points p_n lie in the g_0-geodesic ball $B^{g_0}(q_0, R)$ of radius R around q_0. Since g_0 is complete, this ball is relatively compact, so $\{p_n\}$ has a convergent subsequence. This proves that N is compact.

The bbc property follows directly from the definition as $f = 1$ and $g_t = A_t^* g_0$ is complete by the assumption on the eigenvalues of A_t^{-1} before. This completes the proof. $\square$

2.2 Spinors on Lorentzian cylinders

In this section we describe spinors on a Lorentzian cylinder. For convenience of the reader we first recall some basic facts about Lorentzian spin geometry, thereby fixing our notations. For details we refer to [2] and [1].
Let $(\mathbb{R}^{1,n}, \eta)$ be the $(n+1)$-dimensional Minkowski space with the inner product

$$\eta(x,y) = -x_0 y_0 + x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

where $x = (x_0, x_1, \ldots, x_n)$, $y = (y_0, y_1, \ldots, y_n)$. We fix the standard isometric embedding of the Euclidian space $(\mathbb{R}^n, \epsilon)$ into the Minkowski space

$$\begin{array}{cccc} i: & (\mathbb{R}^n, \epsilon) & \longrightarrow & (\mathbb{R}^{1,n}, \eta) \\ & x & \longmapsto & (0, x) \end{array}$$

with the timelike normal vector $e_0 = (1, 0, \ldots, 0)$.
We denote by $Cl_n := Cliff(\mathbb{R}^n, \epsilon)$ and $Cl_{1,n} := Cliff(\mathbb{R}^{1,n}, \eta)$ the Clifford algebras of the Euclidian space and the Minkowski space, respectively. Since the complex linear map

$$\begin{array}{cccc} \beta: & (\mathbb{C}^n, \epsilon^{\mathbb{C}}) & \longrightarrow & Cliff^0(\mathbb{C}^{1,n}, \eta^{\mathbb{C}}) \\ & x & \longmapsto & ie_0 \cdot x \end{array}$$

satisfies

$$\beta(x) \cdot \beta(x) = -\epsilon^{\mathbb{C}}(x,x) \cdot 1,$$

β induces an isomorphism of the complexified Clifford algebras

$$\tau : Cl_n^{\mathbb{C}} \longrightarrow (Cl_{1,n}^0)^{\mathbb{C}}.$$

Now, let us consider the usual Spin-representation of $Cl_{1,n}$ on the spinor modul $\Delta_{1,n}$

$$\rho_{1,n} : Cl_{1,n} \longrightarrow GL(\Delta_{1,n})$$

and let denote by

$$\rho_{1,n}^{\pm} : Cl_{1,n} \longrightarrow GL(\Delta_{1,n}^{\pm})$$

its positive and negative parts in case of odd n.
If n is even, we consider the action of the Clifford algebra Cl_n on the space $\Delta_{1,n}$ given by

$$\begin{array}{cccccccc} \kappa := \rho_{1,n} \circ \tau : & Cl_n & \longrightarrow & (Cl_{1,n}^0)^{\mathbb{C}} & \longrightarrow & GL(\Delta_{1,n}) & & \\ & x & \longmapsto & ie_0 \cdot x & \longmapsto & \rho_{1,n}(i\, e_0 \cdot x) & , & x \in \mathbb{R}^n \end{array}$$

which is a realization of the unique irreducible complex representation of Cl_n. Let us denote by Δ_n the $Spin(n)$-representation

$$\Delta_n := (\Delta_{1,n}, \kappa|_{Spin(n)}). \tag{1}$$

Then as $Spin(n)$-representations Δ_n and $(\Delta_{1,n}, \rho_{1,n}|_{Spin(n)})$ are isomorphic and the Clifford product under this isomorphism is identified by

$$x \star u \in \Delta_n \longmapsto i\ e_0 \bullet x \bullet u \in \Delta_{1,n} \qquad x \in \mathbb{R}^n,\ u \in \Delta_n, \tag{2}$$

where $x \star u := \kappa(x)u$ denotes the Clifford product on Δ_n and $x \bullet u := \rho_{1,n}(x)u$ denotes the Clifford product on $\Delta_{1,n}$.
If n is odd, the actions of Cl_n on $\Delta^{\pm}_{1,n}$ given by

$$\begin{array}{rcccccl} \kappa^{\pm} := \rho^{\pm}_{1,n} \circ \tau : & Cl_n & \longrightarrow & (Cl^0_{1,n})^{\mathbb{C}} & \longrightarrow & GL(\Delta^{\pm}_{1,n}) & \\ & x & \longmapsto & ie_0 \cdot x & \longmapsto & \rho^{\pm}_{1,n}(i\ e_0 \cdot x) & , \qquad x \in \mathbb{R}^n \end{array}$$

realize the two non-equivalent irreducible complex representations of Cl_n.
We denote by Δ_n and $\hat{\Delta}_n$ the $Spin(n)$-representations

$$\begin{aligned} \Delta_n &:= (\Delta^{+}_{1,n}, \kappa^{+}|_{Spin(n)}) \\ \hat{\Delta}_n &:= (\Delta^{-}_{1,n}, \kappa^{-}|_{Spin(n)}) \end{aligned} \tag{3}$$

The $Spin(n)$-representations (Δ_n, κ^+) and $(\Delta^+_{1,n}, \rho^+_{1,n})$ and the $Spin(n)$-representations $(\hat{\Delta}_n, \kappa^-)$ and $(\Delta^-_{1,n}, \rho^-_{1,n})$ are isomorphic, whereby the Clifford product is identified by

$$x \star u \ := \ \kappa_+(x)u \in \Delta_n \longmapsto i\ e_0 \bullet x \bullet u \in \Delta^+_{1,n} \qquad x \in \mathbb{R}^n,\ u \in \Delta_n, \tag{4}$$

$$x \star v \ := \ \kappa_-(x)v \in \hat{\Delta}_n \longmapsto i\ e_0 \bullet x \bullet v \in \Delta^-_{1,n} \qquad x \in \mathbb{R}^n,\ v \in \hat{\Delta}_n. \tag{5}$$

Furthermore, the linear isomorphism

$$\begin{array}{rccc} \phi : & \Delta_n = \Delta^+_{1,n} & \longrightarrow & \hat{\Delta}_n = \Delta^-_{1,n} \\ & u & \longmapsto & \hat{u} := e_0 \bullet u \end{array}$$

is an isomorphism of the $Spin(n)$-representations and the Clifford product on Δ_n and $\hat{\Delta}_n$ satisfies

$$\widehat{x \star u} = -x \star \hat{u}\ , \qquad x \in \mathbb{R}^n,\ u \in \Delta_n = \Delta^+_{1,n}. \tag{6}$$

On the Lorentzian spinor modul $\Delta_{1,n}$ we have two hermitian products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_0$ which are related by

$$\langle u, v \rangle_1 = \langle e_0 \bullet u, v \rangle_0\ , \qquad u, v \in \Delta_{1,n}. \tag{7}$$

The inner product $\langle \cdot, \cdot \rangle_1$ is $Spin_0(1,n)$-invariant and the Clifford product $x \bullet$ is symmetric for all vectors $x \in \mathbb{R}^{1,n}$. The inner product $\langle \cdot, \cdot \rangle_0$ is $Spin(n)$-invariant and the Clifford product $x \star$ is skew-symmetric for all vectors $x \in \mathbb{R}^n$.

Now, let (M, g_0) be a n-dimensional Riemannian manifold and let $(C = (a,b) \times M, g = -dt^2 + g_t)$ be a Lorentzian cylinder over (M, g_0). In the following we denote by ν the timelike unit normal field on C given by

$$\nu(t,x) := \frac{\partial}{\partial t}(x,t).$$

Now, let us assume that (M, g_0) is a spin manifold with the $SO(n)$-frame bundle P_M, the spin structure (Q_M, f_M) and the corresponding spinor bundles

$$\begin{array}{rcll} S_M & := & Q_M \times_{Spin(n)} \Delta_n & \\ \hat{S}_M & := & Q_M \times_{Spin(n)} \hat{\Delta}_n & \quad \text{if } n \text{ is odd} . \end{array}$$

The spin structure (Q_M, f_M) induces a spin structure on the Lorentzian cylinder (C, g) in a canonical way. To explain that, consider $\gamma_x(t) := (t, x)$ which is a timelike geodesic through $(0, x)$ and let us denote by

$$\tau^t_{\gamma_x} : T_{(0,x)}(\{0\} \times M) \longrightarrow T_{(t,x)}(\{t\} \times M)$$

the parallel displacement in (C, g) along γ_x with respect to the Levi-Civita connection of g and by $\pi : C \longrightarrow M$ the projection $\pi(t, x) = x$. Then the $SO(n)$-principal bundle

$$\hat{P} := \Big\{(\nu(t,x), \tau^t_{\gamma_x}(s_1), \ldots, \tau^t_{\gamma_x}(s_n)) \mid \ x \in M,\ (s_1, \ldots, s_n) \text{ positively oriented ON-basis in } (T_xM, (g_0)_x)\Big\}$$

is a $SO(n)$-reduction of the $SO_0(1, n)$-principal bundle P_C of oriented and time-oriented frames on (C, g) with $\pi^* P_M \simeq \hat{P}$. Therefore the $SO_0(1, n)$-frame bundle of (C, g) can be described as

$$P_C = \pi^*(P_M \times_{SO(n)} SO_0(1, n))$$

and the following pair (Q_C, f_C) is a spin structure of (C, g)

$$Q_C := \pi^*(Q_M \times_{Spin(n)} Spin_0(1, n))$$

$$\begin{array}{rccc} f_C : & Q_C & \longrightarrow & P_C \\ & [q, A] & \longmapsto & [f_M(q), \lambda(A)] \end{array}$$

where $\lambda : Spin_0(1, n) \longrightarrow SO_0(1, n)$ denotes the usual 2-fold covering.
Using (1) and (3) we obtain the following identification for the spinor bundles S_C of (C, g) and S_M of (M, g_0). For even n hold

$$\begin{array}{rcl} S_C & := & Q_C \times_{(Spin_0(1,n),\rho)} \Delta_{1,n} \simeq \pi^*(Q_M \times_{(Spin(n),\kappa)} \Delta_n) \\ & = & \pi^* S_M \end{array} \tag{8}$$

and for odd n

$$\begin{array}{rcl} S_C & = & S^+_C \oplus S^-_C \\ & = & Q_C \times_{(Spin_0(1,n),\rho^+)} \Delta^+_{1,n} \ \oplus\ Q_C \times_{(Spin_0(1,n),\rho^-)} \Delta^-_{1,n} \\ & \simeq & \pi^*\Big(Q_M \times_{(Spin(n),\kappa^+)} \Delta_n \ \oplus\ Q_M \times_{(Spin(n),\kappa^-)} \hat{\Delta}_n\Big) \\ & = & \pi^* S_M \oplus \pi^* \hat{S}_M \end{array} \tag{9}$$

Hence, for even n, any spinor field $\psi \in \Gamma(S_C)$ on the cylinder can be understood as an 1-parameter family of spinors $t \in (a, b) \mapsto \psi(t, \cdot) \in \Gamma(S_M)$ on the manifold M. In case of odd n, any spinor field in $\Gamma(S^+_C)$ is a t-parameter family of spinor fields in $\Gamma(S_M)$, whereas any spinor field in $\Gamma(S^-_C)$ can be understood as t-parameter family of spinors in $\Gamma(\hat{S}_M)$.

Now, for any vector field X and any spinor field φ on (M, g_0) we denote by $\tilde{X}$ and $\widetilde{\varphi}$ the vector field and the spinor field on (C, g) arising from X and φ by parallel displacement along the geodesics γ_x

$$\tilde{X}(t,x) := \tau^t_{\gamma_x}(X(x)) \quad , \quad \widetilde{\varphi}(t,x) := \tau^t_{\gamma_x}(\varphi(x)).$$

Because of (2) and (4) and the Clifford multiplication satisfies

$$\widetilde{X \star \varphi} = i\, \nu \bullet \tilde{X} \bullet \widetilde{\varphi}. \tag{10}$$

Due to its invariance properties, the hermitian products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_0$ on the spinor modul $\Delta_{1,n}$ induce hermitian inner products on the spinor bundel S_C which are related by

$$\langle \psi_1, \psi_2 \rangle_1 = \langle \nu \bullet \psi_1, \psi_2 \rangle_0 \,, \qquad \psi_1, \psi_2 \in \Gamma(S_C). \tag{11}$$

For any spinor field $\psi \in \Gamma(S_C)$ and $\varphi \in \Gamma(S_M)$ we define the associated Dirac current V_ψ on (C, g) and W_φ on (M, g_0), respectively, by

$$g(V_\psi, Y) \;:=\; -\langle Y \bullet \psi, \psi \rangle_1 \qquad \forall Y \in \mathcal{X}(C) \tag{12}$$

$$g_0(W_\varphi, X) \;:=\; i \langle X \star \varphi, \varphi \rangle_0 \qquad \forall X \in \mathcal{X}(M) \tag{13}$$

The Dirac currents satisfy the following conditions, which are easily to verify using (2), (4), (12) and (13)

Proposition 3 *1. The Dirac current V_ψ of a nowhere vanishing spinor field ψ on the Lorentzian cylinder (C, g) is a causal and future directed vector field.*

2. Let φ be a spinor field on (M, g_0) and $\widetilde{\varphi}$ its parallel extension to the cylinder (C, g) along the geodesics γ_x. Then for the Dirac currents hold

$$V_{\widetilde{\varphi}} = \|\widetilde{\varphi}\|_0^2 \, \nu + \widetilde{W_\varphi} \tag{14}$$

□

Comparing the Levi-Civita connections of the cylinder (C, g) and of the level sets $(M_t := \{t\} \times M, g_t)$ one obtains the following relation between the spinor derivatives:

Proposition 4 *(cf. [1]) Let X be a vector field and φ a spinor field on (M, g_0). Then for the parallel transported spinor field $\widetilde{\varphi}$ on (C, g) hold $\nabla^C_\nu \widetilde{\varphi} = 0$ by definition and*

$$\nabla^C_{\tilde{X}} \widetilde{\varphi} \;=\; \nabla^{M_t}_{\tilde{X}} \widetilde{\varphi} + \frac{1}{2} \nu \bullet S_t(\tilde{X})) \bullet \widetilde{\varphi} \qquad \text{on } M_t \tag{15}$$

where $S_t(Y) := -\nabla^C_Y \nu$ is the Weingarten map of the submanifold $M_t \subset C$. □

2.3 Codazzi spinors and special Lorentzian cylinders

If B is an invertible endomorphism field and h a pseudo-Riemannian metric on M, we denote by B^*h the induced metric

$$B^*h(X,Y) := h(BX,BY).$$

In the following we will consider a special example of Lorentzian cylinders.
Let A be a symmetric uniformally bounded endomorphism field. We denote by $\mu_+(A)$ the supremum of the positive eigenvalues of A or zero if all eigenvalues of A are nonpositive and by $\mu_-(A)$ the infimum of the negative eigenvalues of A or zero if all eigenvalues of A are nonnegative. Now, set $a := (2\mu_-(A))^{-1}$ and $b := (2\mu_+(A))^{-1}$. We denote by $C(M;A)$ the Lorentzian cylinder

$$C(M;A) := (a,b)\times M \ , \quad g := -dt^2 + g_t := -dt^2 + (\mathbf{1} - 2tA)^* g_0$$

call it *Lorentzian cylinder over M induced by A.*

Definition: Let (M, g_0) be a Riemannian spin manifold with spinor bundle S_M. An *imaginary Codazzi spinor* on M is a spinor field $\psi \in \Gamma(S_M)$ which satisfies the equation

$$\nabla_X \psi = i \cdot A(X) \star \psi, \tag{16}$$

where A is a Codazzi tensor on (M, g_0), t.m. A is a symmetric endomorphism field which obeys the Codazzi equation

$$(\nabla^M_X A)(Y) = (\nabla^M_Y A)(X)$$

for all vector fields X, Y on M. The equation (16) can be interpreted as the one of parallelity w.r.t. the (non-metric) connection $\nabla - iA$.

The notion of imaginary Codazzi spinors generalizes the case of imaginary Killing spinors with Killing number $i\lambda$, which are Codazzi spinors with $A = \lambda\mathbf{1}$.
Note that the Codazzi-tensor A is uniquely defined by its Codazzi spinor (16) since

$$g_0(A(X),Y) = -\frac{1}{2}\, Im\Big(\langle X \star \nabla^M_Y \psi + Y \star \nabla^M_X \psi, \psi \rangle_0 \Big) \cdot \|\psi\|_0^{-2}.$$

Furthermore, the existence of an imaginary Codazzi spinor implies the following curvature constraint on (M, g_0)

Proposition 5 *Let (M, g_0) be a Riemannian manifold with non-trivial imaginary Codazzi spinor ψ to the Codazzi tensor A. Then the Ricci curvature of (M, g_0) satisfies*

$$Ric^M(X) = 4A^2(X) - 4(trA)\cdot A(X).$$

Proof. The Codazzi equation for A implies

$$R^{S_M}(X,Y)\psi = \Big(A(X) \star A(Y) - A(Y) \star A(X) \Big) \star \psi.$$

Hence

$$Ric(X) \star \psi = -2 \sum_{k=1}^{n} s_k \star R^{S_M}(X, s_k)\psi = (4A^2(X) - 4tr(A)\cdot A(X)) \star \psi.$$

Since ψ is nontrivial, it vanishes nowhere. Therefore, the vectors in front of ψ in the latter formula on both sides coincide. □

In [1] the authors study *real* Codazzi spinors and cylinders with *spacelike* cylinder axis and show, that Codazzi spinors on M can be extended to parallel spinors on the cylinder. The same statement is true in case of imaginary Codazzi spinors and Lorentzian cylinders with the completely analogous proof:

Proposition 6 *Let (M, g_0) be a Riemannian spin manifold carrying an imaginary Codazzi spinor $\psi \in \Gamma(S_M)$ with uniformally bounded Codazzi tensor A and let $C := C(M; A)$ be the Lorentzian cylinder over (M, g_0) induced by A with its canonical spin structure. Then the ν-parallel extension $\widetilde{\psi} \in \Gamma(S_C^{(+)})$ of ψ is a parallel spinor field on the cylinder $C(M; A)$. Conversely, the restriction of any parallel spinor $\phi \in \Gamma(S_C^{(+)})$ of the cylinder to M_0 is a Codazzi spinor to the Codazzi tensor A.*

Proof. We only recall the main steps of the proof and refer for details to [1].
First, it is easy to see that $2A$ is the Weingarten map of the submanifold $M_0 = \{0\} \times M \subset C(M; A) =: C$. Then from (10) and (15) follows that for any vector field X on M_0

$$\nabla^C_X \widetilde{\psi} = \nabla^{M_0}_X \widetilde{\psi} + \nu \bullet A(X) \bullet \widetilde{\psi} = \nabla^M_X \psi - iA(X) \star \psi. \tag{17}$$

Since ψ is an imaginary Codazzi spinor, we obtain $\nabla^C_X \widetilde{\psi} = 0$ for all vectors X tangential to M_0. The Codazzi equation for A implies that ν is in the kernel of the curvature endomorphism, i.e. $R^C(\nu, \cdot) = 0$. Since $[\nu, X] = 0$ and $\nabla^C_\nu \widetilde{\psi} = 0$ it follows

$$\nabla^C_\nu \nabla^C_X \widetilde{\psi} = \frac{1}{2} R^C(\nu, X) \bullet \widetilde{\psi} = 0$$

where X is the lift of a vector field X of M to C. But this means that the spinor field $\nabla^C_X \widetilde{\psi}$ is parallel along all geodesics γ_x. As this spinor field vanishes in the point $\gamma_x(0) = (0, x)$, it vanishes everywhere on C. Hence $\widetilde{\psi}$ is parallel on C. On the other hand, any parallel spinor ϕ on the cylinder is the parallel extension of its restriction to M_0 along the curves γ_x. (17) shows that this restriction is an imaginary Codazzi tensor. □

For a spinor field $\varphi \in \Gamma(S_M)$ we consider the the real subbundle $E_\varphi := TM \star \varphi \subset S_M$ and denote by $dist_\varphi$ the pointwise distance between $i\varphi$ and E_φ with respect to the real inner product $Re\langle \cdot, \cdot \rangle_0$ on S_M.
In order to analyze the causal type of the Dirac current $V_{\widetilde{\psi}}$ of the parallel extension $\widetilde{\psi}$ of a Codazzi spinor ψ we will use the following Lemma

Lemma 1 *Let (M, g_0) be a Riemannian manifold with an endomorphism field B and let $\varphi \in \Gamma(S_M)$ be a spinor field such that*

$$\nabla^M_X \varphi = i\, B(X) \star \varphi. \tag{18}$$

We denote by W_φ the Dirac current of φ and by q_φ the function

$$q_\varphi := \|\varphi\|_0^4 - g_0(W_\varphi, W_\varphi). \tag{19}$$

Then q_φ is constant and non-negative and given by the distance function

$$q_\varphi = dist^2_\varphi \cdot \|\varphi\|^2_0.$$

Proof. Using formula (13) for the Dirac current W_φ and formula (18) for the spinor φ we obtain

$$\nabla^M_X W_\varphi = 2\|\varphi\|_0^2 \cdot B(X).$$

From that follows

$$\begin{aligned} X(q_\varphi) &= 4\|\varphi\|_0^2\, i\, \langle B(X)\star\varphi,\varphi\rangle_0 - 2g_0(\nabla_X W_\varphi, W_\varphi) \\ &= 4\|\varphi\|_0^2 g_0(B(X),W_\varphi) - 4\|\varphi\|_0^2 g_0(B(X),W_\varphi) = 0. \end{aligned}$$

If $(s_1,\dots,s_n)$ is an ON-basis on M, then $(\|\varphi\|^{-1}\, s_j\star\varphi \mid j=1,\dots,n)$ is an ON-basis in the real vector space E_φ. Hence

$$\begin{aligned} dist_\varphi^2 &= \|i\varphi\|_0^2 - \|proj_{E_\varphi}(i\varphi)\|_0^2 \\ &= \|\varphi\|_0^2 - \|\sum_{j=1}^n \|\varphi\|_0^{-2}\cdot\langle i\varphi, s_j\star\varphi\rangle_0 \cdot s_j\star\varphi\|_0^2 \\ &= \|\varphi\|_0^2 - g_0(W_\varphi,W_\varphi)\cdot\|\varphi\|_0^{-2} \;=\; q_\varphi\cdot\|\varphi\|_0^{-2}. \end{aligned}$$

□

Now, we describe the causal type of the Dirac current of a parallel spinor on the cylinder which is induced by an imaginary Codazzi spinor.

Proposition 7 *Let ψ be a non-vanishing imaginary Codazzi spinor on (M,g_0) with uniformally bounded Codazzi tensor A and let $\widetilde{\psi}$ be its ν-parallel extension to the cylinder $C(M;A)$. If $dist_\psi = 0$, the Dirac current $V_{\widetilde{\psi}}$ of $\widetilde{\psi}$ is parallel and lightlike. If $dist_\psi > 0$, the Dirac current $V_{\widetilde{\psi}}$ is parallel and timelike.*

Proof. Since $\widetilde{\psi}$ is parallel, the Dirac current $V_{\widetilde{\psi}}$ is parallel as well. ψ is non-trivial, hence the length $\|\psi\|_0^2$ has no zeros. From formula (14) follows

$$\begin{aligned} g(V_{\widetilde{\psi}},V_{\widetilde{\psi}}) &= -\|\widetilde{\psi}\|_0^4 + g(\tilde{W}_\psi,\tilde{W}_\psi) = -\|\psi\|^4 + g_0(W_\psi,W_\psi) = -q_\psi \\ &= -\|\psi\|_0^2\cdot dist_\psi \end{aligned}$$

which proves the statement. □

We remark, that for any spinor φ on a 3- and on a 5-dimensional manifold the distance $dist_\varphi$ is zero (cf. [5], p. 89).

3 Codazzi spinors with invertible Codazzi tensors

3.1 The structure theorem

Let us denote by d^∇ the exterior derivative induced by a covariant derivative ∇. In particular, for any endomorphism field A

$$(d^\nabla A)(X,Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X).$$

Hence A is a Codazzi tensor with respect to ∇ if and only if $d^\nabla A = 0$.
In the following we want to examine the case of invertible Codazzi tensors more closely. First we remark

Proposition 8 *([8]) Let A be an invertible Codazzi tensor on (M, g_0) and let ∇^{g_0} and $\nabla^{A^*g_0}$ denote the Levi-Civita connections of g_0 and of the induced metric A^*g_0. Then*

$$\nabla_X^{A^*g_0} = A^{-1} \circ \nabla_X^{g_0} \circ A\,, \qquad \forall\, X \in \mathcal{X}(M)$$

and A^{-1} is a Codazzi tensor w.r.t. A^*g_0. In particular, the holonomy groups of g_0 and A^*g_0 are conjugated

$$Hol_x(M, A^*g_0) = A_x^{-1} \circ Hol_x(M, g_0) \circ A_x$$

and for the curvature hold

$$\begin{aligned} R^{A^*g_0}(X,Y,Z,V) &= R^{g_0}(X,Y,AZ,AV) \\ Ric^{g_0} \circ A &= A \circ Ric^{g_0}. \end{aligned}$$

□

There is an obvious identification between the spin structures of the manifolds (M, g_0) and (M, A^*g_0). Let us denote by

$$\Phi_A : \ \varphi \in S_M = S_{(M,g_0)} \longrightarrow \overline{\varphi} \in \overline{S}_M := S_{(M,A^*g_0)}$$

the corresponding isomorphism between the spinor bundles. Then for the Clifford product and the spinor derivatives $\nabla = \nabla^{g_0}$ and $\overline{\nabla} = \nabla^{A^*g_0}$ hold

$$\overline{X \star \varphi} = A^{-1}(X) \star \overline{\varphi} \qquad \text{and} \qquad \overline{\nabla_X \varphi} = \overline{\nabla}_X \overline{\varphi}. \tag{20}$$

Furthermore, we obtain

$$\langle \varphi, \psi \rangle_0 = \langle \overline{\varphi}, \overline{\psi} \rangle_0 \qquad \text{and} \qquad W_{\overline{\varphi}} = A^{-1}(W_\varphi) \tag{21}$$

which shows that $q_\varphi = q_{\overline{\varphi}}$ for the functions q_φ defined in (19). Hence we obtain the following Proposition:

Proposition 9 *Let φ be an imaginary Codazzi spinor on a Riemannian manifold (M, g_0) with invertible Codazzi tensor A. Then the corresponding spinor $\overline{\varphi}$ on (M, A^*g_0) is a Killing spinor with Killing number i. Vice versa, given a Riemannian manifold (M, g_0) with a Killing spinor ϕ to the Killing number ia for some real number a and with an invertible Codazzi tensor A, on the Riemannian manifold (M, A^*g_0) the spinor field $\Phi_A(\phi) = \overline{\phi}$ is a Codazzi spinor with Codazzi tensor aA^{-1}. Likewise, Φ_A maps parallel spinor fields (if they exist) to parallel spinor fields.* □

Thus we can use the structure results the first author proved for manifolds with imaginary Killing spinors.

Proposition 10 *([3], [4]) (M, g_0) is a complete connected Riemannian spin manifold with non-vanishing imaginary Killing spinor to the Killing number $i\mu$ if and only if it is isometric to a warped product*

$$(\mathbb{R} \times F, ds^2 + e^{-4\mu s} g_F)$$

where (F, g_F) is a complete connected spin manifold with non-vanishing parallel spinor.
□

Let B a symmetric, positive definite endomorphism field on (M, g_0) whose eigenvalues are uniformally bounded away from zero, t.m. with $0 < \|B^{-1}\| \leq c < \infty$ on all of M. Then the metric $B^* g_0$ is complete if g_0 is complete. From Proposition 9, Proposition 10 and Proposition 7 we obtain the following structure of a Riemannian manifold with imaginary Codazzi spinors.

Theorem 1 *Let (M, g_0) be a complete connected Riemannian spin manifold with non-vanishing imaginary Codazzi spinor to a Codazzi tensor A whose eigenvalues are uniformally bounded away from zero. Then $(M, A^* g_0)$ is isometric to a warped product*

$$(\mathbb{R} \times F,\ g_{wp} := ds^2 + e^{-4s} g_F) \qquad (*)$$

where (F, g_F) is a complete connected spin manifold with non-vanishing parallel spinor. Furthermore, A^{-1} is a Codazzi tensor on the warped product (*). □

3.2 Codazzi tensors on warped products

Theorem 1 shows how we can obtain Riemannian manifolds with Codazzi spinors. For that we have to find Codazzi tensors on warped products of the form

$$M = \mathbb{R} \times_f F := (\mathbb{R} \times F, g_M = ds^2 + f^2(s) g_F).$$

Let us now address to this question. We decompose $T_{(s,x)} M \simeq \mathbb{R}\partial_s \oplus T_x F$ and write a symmetric $(1,1)$-tensor field H on M with respect to this decomposition as

$$H = H(b, D, E) = \begin{pmatrix} b \cdot Id & \widetilde{D} \\ D & E \end{pmatrix}$$

where b is a real function on $\mathbb{R} \times F$, E is a s-parametrized family of symmetric endomorphism fields on (F, g_F), D an endomorphism field from $\mathbb{R}\partial_s$ to TF, and

$$\widetilde{D} V = f^2 \cdot g_F(V, D(\partial_s))\, \partial_s$$

for all vector fields V on F.

Let us call the tensor field $H = H(b, 0, E)$ *simple* iff $E(s) = K(s) \cdot Id_F$ for all $s \in \mathbb{R}$.

Proposition 11 *The endomorphism field $H(b, D, E)$ is a Codazzi tensor on the warped product $M = \mathbb{R} \times_f F$ if and only if for all vector fields V and W on F hold*

$$\begin{aligned} \nabla^F_V(D(\partial_s)) &= \dot{E}(V) + \frac{\dot{f}}{f} E(V) - b \frac{\dot{f}}{f} V && (22) \\ grad^F b &= 3 f \dot{f} D(\partial_s) + f^2 \dot{D}(\partial_s) && (23) \\ (d^{\nabla^F} E)(V, W) &= f \dot{f} \Big(g_F(V, D(\partial_s)) W - g_F(W, D(\partial_s)) V \Big) && (24) \end{aligned}$$

In particular, the s-parameter family $D(\partial_s)$ of vector fields on F satisfies

$$R^F(V, W) D(\partial_s) = (f \ddot{f} - \dot{f}^2) \Big(g_F(V, D(\partial_s)) W - g_F(W, D(\partial_s)) V \Big). \qquad (25)$$

Here $\dot{}$ denotes the derivative with respect to the parameter s.

Proof. We rewrite $d^{\nabla^M} H = 0$ in conditions for b, D and E. The covariant derivative on the warped product $M = \mathbb{R} \times_f F$ is given by

$$\begin{aligned}
\nabla^M_{\partial_s} \partial_s &= 0 \\
\nabla^M_{\partial_s} V &= \nabla^M_V \partial_s = \frac{\dot{f}}{f} V \\
\nabla^M_V W &= -f\dot{f} g_F(V,W)\partial_s + \nabla^F_V W.
\end{aligned}$$

(Cf. [21], p. 206)). Using these formulas, one easily sees that the formulas (22) and (23) are equivalent to $(d^{\nabla^M} H)(\partial_s, V) = 0$. The condition $(d^{\nabla^M} H)(V,W) = 0$ is equivalent to (24) and

$$g_F(W, \nabla^F_V D(\partial_s)) = g_F(V, \nabla^F_W D(\partial_s)),$$

where the latter formula follows already from (22). Now, if we differentiate (22) again with respect to ∇^F_W and insert (23) and (24) we obtain (25). □

Recall that a Codazzi tensor is called *trivial* if it is a constant multiple of the Identity.

Corollary 1 *1. On every warped product $M = \mathbb{R} \times_f F$ there is a nontrivial invertible Codazzi tensor. Explicitly, if T is a (possibly trivial) Codazzi tensor on (F, g_F), then we obtain a Codazzi tensor on M by $H = H(b, 0, E)$ where b is a function which depends only on s and E is given by*

$$E(s) = \frac{1}{f}\Big(T + \int_0^s b(\sigma)\dot{f}(\sigma) d\sigma \cdot Id_F\Big) \tag{26}$$

2. Let $f(s) = e^{k \cdot s}$. If there is a non-simple Codazzi tensor on M then there is a nontrivial Codazzi tensor on F or a nonzero, parallel or homothetic vector field on F.

Proof. It is easy to check, using (22), (23) and (24), that the tensor field $H(b, 0, E)$, where $b = b(s)$ is a real function and E is given by (26), is a Codazzi tensor on $M = \mathbb{R} \times_f F$, for every Codazzi tensor T on F. If we choose for example b so that $b\dot{f} > 0$ and $T = k \cdot Id_F$ with $k > 0$, then $H(b, 0, T)$ is invertible.
Now let $H(b, D, E)$ be a Codazzi tensor on $M = \mathbb{R} \times_f F$, first with a general warping function f. Let us first consider the case that $D \equiv 0$. Then formula (23) shows, that the function b depends only on s. From (24) follows, that E is a Codazzi tensor on (F, g_F). From (22) we obtain the following differential equation in the space of Codazzi tensors on (F, g_F)

$$(\dot{fE}) = \dot{f}E + f\dot{E} = b\dot{f} Id_F.$$

Hence E is given by formula (26), where T is a Codazzi tensor on (F, g_F). H is simple exactly if T is trivial.
Now, let $D \neq 0$. We set

$$\tilde{E} := (f\ddot{f} - \dot{f}^2)E - f\dot{f}\nabla^F D(\partial_s).$$

From (25) follows

$$\left(d^{\nabla^F} d^{\nabla^F} D(\partial_s)\right)(V,W) = R^F(V,W)D(\partial_s) = (f\ddot{f} - \dot{f}^2)\Big(g_F(V, D(\partial_s))W - g_F(W, D(\partial_s))V\Big).$$

Using (24) we obtain

$$
\begin{aligned}
(d^{\nabla^F}\tilde{E})(V,W) &= (f\ddot{f}-\dot{f}^2)(d^{\nabla^F}E)(V,W) - f\dot{f}\,(d^{\nabla^F}d^{\nabla^F}D(\partial_s))\,(V,W)\\
&= (f\ddot{f}-\dot{f}^2)\Big((d^{\nabla^F}E)(V,W) - f\dot{f}(g_F(V,D(\partial_s))W - g_F(W,D(\partial_s))V)\Big)\\
&= 0.
\end{aligned}
$$

Thus $\tilde{E}$ is a Codazzi tensor on (F,g_F).
In our case, for $f = e^{k\cdot s}$, we have $f\ddot{f} - \dot{f}^2 \equiv 0$. Therefore, $\tilde{E} = f\dot{f}\nabla^F D(\partial_s)$, thus $\nabla^F D(\partial_s)$ is a Codazzi tensor on (F,g_F). If this Codazzi tensor is trivial, t.m. $\nabla^F_{\cdot} D(\partial_s) = \lambda Id_F$, then $D(\partial_s)$ is a family of parallel vector fields on F (if $\lambda = 0$) or of homothetic vector fields on F (if $\lambda \neq 0$). □

Corollary 2 *Let $M = \mathbb{R}\times_f F$ be a warped product with warping function $f(s) = e^{-2s}$ and let T be a Codazzi tensor on (F,g_F) whose eigenvalues are uniformally bounded from below. Then the warped product M admits a Codazzi tensor H with eigenvalues uniformally bounded away from zero ($\|H^{-1}\| \leq c < \infty$ on all of M).*

Proof. Let all eigenvalues of T be greater then $k \in \mathbb{R}$. We choose a strictly increasing function $h \in C^\infty(\mathbb{R})$ with $h(s) < \frac{k}{2}$. Let $b(s) := e^{2s}\dot{h}(s)$. Then all eigenvalues of the Codazzi tensor $H = H(b,0,E)$, where E is given by (26), are nonnegative. Hence the norm of the Codazzi tensor $H + cId_M$ for a positive constant c is bounded from below by c. □

3.3 The holonomy and causal properties of the special Lorentzian cylinder

Theorem 1 and Corollary 2 provide a construction principle for Riemannian manifolds with Codazzi spinors and for Lorentzian manifolds with special holonomy. We start with a connected Riemannian manifold (F,g_F) which admits a non-vanishing parallel spinor and a Codazzi tensor with eigenvalues uniformally bounded from below. Then the warped product $(M := \mathbb{R}\times F,\ g_{wp} := ds^2 + e^{-4s}g_F)$ admits imaginary Killing spinors to the Killing number i and a Codazzi tensor H with positive eigenvalues uniformally bounded away from zero. The Riemannian manifold $(M, g_0 := H^* g_{wp})$ admits imaginary Codazzi spinors for the Codazzi tensor H^{-1}.
In the following we will denote the Lorentzian cylinder constructed in this way out of (F,g_F) and H by $C[F;H]$. By construction

$$C[F;H] := (a,b)\times M = (a,b)\times\mathbb{R}\times F$$

where a is the half of the supremum of all negative eigenvalues of H or $-\infty$ if all eigenvalues of H are positive and b is the half of the infimum of all positive eigenvalues of H or $+\infty$ if all eigenvalues of H are negative. The metric of $C[F;H]$ is given by

$$g_C := -dt^2 + g_t = -dt^2 + (H - 2t\mathbf{1})^*(ds^2 + e^{-4s}g_F).$$

In the following we will denote

$$
\begin{aligned}
A &:= H^{-1}\\
H_t &:= (H - 2tId_M)
\end{aligned}
$$

First, let us start with the causality property of the cylinder $C[F;H]$.

Theorem 2 *If the Riemannian manifold (F, g_F) is complete, then the cylinder $C[F;H]$ is globally hyperbolic and even bbc.*

Proof. Since (F, g_F) is complete, the warped product $(M = \mathbb{R} \times_{e^{-2s}} F, g_{wp})$ is complete as well. The eigenvalues of $(H - 2t\mathbf{1})$ are uniformally bounded away from zero for all $t \in (a, b)$, hence Proposition 2 yields that the Lorentzian cylinder $C[F;H]$ is globally hyperbolic and moreover bbc □

Next we would like to calculate the curvature of $C[F;H]$. For that we note

Lemma 2 *The Weingarten map W_t of the hypersurface $M_t = \{t\} \times M \subset C[F;H]$ is given by*

$$W_t(X) = 2H_t^{-1}(X). \tag{27}$$

The covariant derivative of $C[F;H]$ is

$$\begin{aligned} \nabla^C_X Y &= (H_t^{-1} \circ \nabla^{wp}_X \circ H_t)Y - 2g_{wp}(H_t X, Y)\partial_t & (28)\\ \nabla^C_X \partial_t &= -2H_t^{-1}X & (29)\\ \nabla^C_{\partial_t} X &= -2H_t^{-1}X & (30)\\ \nabla^C_{\partial_t}\partial_t &= 0 & (31) \end{aligned}$$

where X and Y are lifts of vector fields of M.

Proof. For the Weingarten map $W_t(X) = -\nabla^C_X \partial_t$ of M_t we have

$$g_t(W_t(X), Y) = -\frac{1}{2}\dot{g}_t(X, Y).$$

since $g_t = (H - 2tId)^* g_{wp}$ we obtain

$$\dot{g}_t(X, Y) = -4g_{wp}(H_t X, Y) = -4g_t(X, H_t^{-1}Y) = -4g_t(H_t^{-1}X, Y)$$

Hence

$$W_t(X) = 2H_t^{-1}(X).$$

The Gauß decomposition of the covariant derivative gives

$$\begin{aligned} \nabla^C_X Y &= \nabla^{g_t}_X Y - g_t(\nabla^C_X Y, \partial_t)\partial_t = \nabla^{g_t}_X Y - g_t(W_t(X), Y)\partial_t \\ &= \nabla^{g_t}_X Y - 2g_t(H_t^{-1}(X), Y)\partial_t = \nabla^{g_t}_X Y - 2g_{wp}(X, H_t Y)\partial_t \\ &= \nabla^{g_t}_X Y - 2g_{wp}(H_t X, Y)\partial_t. \end{aligned}$$

Then (28) follows from Proposition 8, since H_t is a Codazzi tensor. The equation (31) holds for any cylinder metric and (29) and (30) follow from (27). □

Proposition 12 *The curvature of the Lorentzian cylinder $C[F;H]$ satisfies:*

1. $R^C(X,Y)\partial_t = R^C(X,\partial_t)Y = R^C(X,\partial_t)\partial_t = 0$ *for all vector fields* X, Y *on* C.

2. *If* $X,Y,V \in TM_t$ *then*

$$R^C(X,Y)V = (H_t^{-1} \circ R^{wp}(X,Y) \circ H_t)\,V - 4g_{wp}(X,H_tV)H_t^{-1}Y + 4g_{wp}(Y,H_tV)H_t^{-1}X \tag{32}$$

2. *If the vectors* $X,Y,V \in TM_t$ *are lifts of vectors in* TF *then*

$$R^C(X,Y)H_t^{-1}V = H_t^{-1}R^F(X,Y)V \tag{33}$$

3. *If the vectors* $X,Y \in TM_t$ *that are lifts of vectors in* TF *then*

$$R^C(X,Y)H_t^{-1}(\partial_s) = R^C(\partial_s,Y)H_t^{-1}(\partial_s) = R^C(\partial_s,Y)H_t^{-1}(X) = 0. \tag{34}$$

In particular, $C[F;H]$ is flat if and only if (F,g_F) is flat.

Proof. The first statement is a direct calculation using the Codazzi-Mainardi equation and the Ricatti equation (comp. [1]). The Codazzi-Mainardi equation ([21], p.115) shows that $R^C(X,Y)V$ is tangent to M_t. Using the Gauß equation ([21], p. 100) and the formula (27) for the Weingarten map we obtain

$$g_t(R^C(X,Y)V,W) = \\ = g_t(R^{M_t}(X,Y)V,W) - 4(g_t(H_t^{-1}X,V)g_t(H_t^{-1}Y,W) + 4g_t(H_t^{-1}X,W)g_t(H_t^{-1}Y,V))$$

Since $g_t = H_t^* g_{wp}$ we can apply Proposition 8 which yields

$$\begin{aligned} g_t(R^C(X,Y)V,W) &= H_t^* g_{wp}(R^{H_t^* g_{wp}}(X,Y)V,W) - 4g_{wp}(X,H_tV)g_t(H_t^- Y,W) \\ &\quad +4g_t(H_t^{-1}X,W)g_{wp}(Y,H_tV) \\ &= g_t(H_t^{-1}\,R^{g_{wp}}(X,Y)H_tV,W) - 4g_{wp}(X,H_tV)g_t(H_t^{-1}Y,W) \\ &\quad +4g_{wp}(Y,H_tV)g_t(H_t^{-1}X,W). \end{aligned}$$

Hence

$$R^C(X,Y)V = H_t^{-1} \circ R^{wp}(X,Y) \circ H_t\,V - 4g_{wp}(X,H_tV)H_t^{-1}Y + 4g_{wp}(Y,H_tV)H_t^{-1}X.$$

For a warped product $M = \mathbb{R} \times_f F$ and vector fields U,X,Y which are lifts of vector fields on F, we have

$$R^{wp}(X,Y)U = R^F(X,Y)U + 4g_{wp}(X,U)Y - 4g_{wp}(Y,U)X \tag{35}$$
$$R^{wp}(\partial_s,Y)U = -4f^2 g_F(Y,U)\partial_s \tag{36}$$
$$R^{wp}(\partial_s,Y)\partial_s = 4Y \tag{37}$$
$$R^{wp}(X,Y)\partial_s = 0. \tag{38}$$

(cf. [21], p. 210). This shows formulas (33) and (34). □

Finally we would like to study the holonomy of the cylinder $C[F;H]$. First, let us make a comment on the holonomy of the manifolds $(M = \mathbb{R} \times F\,,\ H_t^* g_{wp})$.

Proposition 13 *The connected component of the holonomy group of the manifolds* $(M = \mathbb{R} \times F\,,\, H_t^* g_{wp})$ *is isomorphic to* $SO(n)$.

Proof. By Proposition 8 it is enough to prove this for (M, g_{wp}). Fix a point $z = (0, x) \in M$ and consider the skew symmetric endomorphism $B := 4\partial_s \wedge Y \subset \mathfrak{so}(T_z M, g_{wp}) \simeq \Lambda^2(T_z M)$. Then formulas (36) and (37) show that

$$\begin{aligned} B(U) &= -4g_F(Y,U)\partial_s \;=\; R_z^{wp}(\partial_s, Y)U \\ B(\partial_s) &= 4Y \;=\; R_z^{wp}(\partial_s, Y)\partial_s \end{aligned}$$

for all $Y, U \in T_x F$. By the Ambrose-Singer Theorem $\partial_s \wedge Y = \frac{1}{4}B = \frac{1}{4}R_z^{wp}(\partial_s, Y) \in \mathfrak{hol}_z(M, g_{wp})$ for all $Y \in T_x F$. Since $[\partial_s \wedge X, \partial_s \wedge Y] = X \wedge Y$ if follows that $\mathfrak{hol}_z(M, g_{wp}) = \mathfrak{so}(T_z M, g_{wp})$. □

We will determine the holonomy group of $C[F; H]$ by calculating the parallel displacement explicitly. For that we use the following Lemmata.

Lemma 3 *The Dirac current of any parallel spinor* $\widetilde{\varphi}$ *on* $C[F; H]$ *that is induced by a parallel spinor on* (F, g_F) *is light-like and given by*

$$V_{\widetilde{\varphi}} = e^{-2s}(\partial_t - \widetilde{A\partial_s})$$

where $\widetilde{X}$ *denotes the parallel displacement of a vector field* X *on* M_0 *along the* t*-lines of the cylinder.*

Proof. In [3] and [4] it is proven, that for any imaginary Killing spinor ψ on (M, g_{wp}), constructed out of a parallel spinor of (F, g_F), hold $q_\psi = 0$. The length and the Dirac current W_ψ of ψ are given by

$$\|\psi(s,x)\|_0^2 = e^{-2s} \quad \text{and} \quad W_\psi = -e^{-2s}\partial_s$$

where we normalize ψ by $1 = \|\psi(0,\cdot)\|_0^2$. Let us denote by $\varphi = \overline{\psi}$ the corresponding Codazzi spinor on $(M, g_0 := H^* g_{wp})$. Then (21) shows that the length and the Dirac current of φ are given by

$$\|\varphi\|_0^2 = e^{-2s} \quad \text{and} \quad W_\varphi = -e^{-2s}H^{-1}(\partial_s).$$

According to Proposition 3 and Proposition 7 the Dirac current of the parallel extended spinor $\widetilde{\varphi}$ on the cylinder $C[F, H]$ induced by φ is parallel and lightlike and given by

$$V_{\widetilde{\varphi}} = e^{-2s}(\partial_t - \widetilde{H^{-1}(\partial_s)}).$$

□

Lemma 4 *Let* Z *be a vector in* TM_0. *Then the parallel displacement* $\widetilde{AZ}$ *of* AZ *along the* t*-lines of the cylinder is given by*

$$\widetilde{AZ} = H_t^{-1}(Z) \tag{39}$$

If V and W be lifts of vector fields of F, then

$$\begin{aligned}
\nabla^C_{\partial_s}\widetilde{A\partial_s} &= -2\partial_t & (40)\\
\nabla^C_{\partial_s}\widetilde{AV} &= -2H_t^{-1}(V) & (41)\\
\nabla^C_V\widetilde{A\partial_s} &= -2H_t^{-1}(V) & (42)\\
\nabla^C_V\widetilde{AW} &= -2e^{-4s}g_F(V,W)\Big(\partial_t - H_t^{-1}(\partial_s)\Big) + H_t^{-1}(\nabla^F_V W) & (43)
\end{aligned}$$

Proof. From Lemma 2 we obtain

$$\nabla^C_{\partial_t}(H_t^{-1}Z) = \nabla^C_{H_t^{-1}Z}\partial_t + [\partial_t, H_t^{-1}Z] = -2H_t^{-2}Z + (H_t^{-1})'Z.$$

Now, from $H_t \circ H_t^{-1} = Id$ follows

$$0 = H_t' \circ H_t^{-1} + H_t \circ (H_t^{-1})' = -2H_t^{-1} + H_t \circ (H_t^{-1})'$$

and therefore

$$(H_t^{-1})' = 2H_t^{-2}.$$

Hence $H_t^{-1}Z$ is parallel along the t-lines.
To prove the formulas for the covariant derivative we use the formulas (cf. [21]) for the covariant derivative of warped products

$$\begin{aligned}
\nabla^{wp}_{\partial_s}\partial_s &= 0\\
\nabla^{wp}_{\partial_s}V &= \nabla^{wp}_V\partial_s = -2V\\
\nabla^{wp}_V W &= 2e^{-4s}g_F(V,W)\partial_s + \nabla^F_V W
\end{aligned}$$

and the formulas (28) and (39). □

Now let us denote by P and Q the following light-like vector fields on $C[F;H]$

$$\begin{aligned}
P &= e^{-2s}(\partial_t - \widetilde{A\partial_s})\\
Q &= \frac{1}{2}e^{2s}(\partial_t + \widetilde{A\partial_s})
\end{aligned}$$

Then $g_C(P;Q) = 1$ and P is parallel (which is clear, since P is by Lemma 4 the Dirac current of a parallel spinor, or can be calculated with the formulas (40) and (42)). The tangent space TC decomposes into the following subspaces

$$TC = \mathbb{R}P \oplus \widetilde{ATF} \oplus \mathbb{R}Q.$$

Proposition 14 *Let $\delta(r) = (t(r), s(r), \gamma(r))$ be a curve in $C = C[F;H]$ starting in $\delta(0) = (0,0,x) \in C$. Then the parallel displacement of $AZ \in AT_xF$ is given by*

$$\tau_{\delta(r)}(AZ) = U(r) = a_{\gamma,Z}(r)P + e^{2s(r)}\widetilde{AY(r)}$$

where $Y(r) = \tau^F_{\gamma|_{[0,r]}}(Z)$ is the parallel displacement of Z along γ in F, and $a_{\gamma,Z}$ is the function determined by

$$\dot{a}_{\gamma,Z}(r) = 2g_F(Y(r),\dot{\gamma}(r)) \quad , \quad a_{\gamma,Z}(0) = 0.$$

Proof. We consider the vector field

$$U(r) = a(r)P + b(r)Q + \widetilde{AZ(r)}.$$

Using (40), (41) and (42) we easily check that

$$\begin{aligned} \nabla^C_{\partial_t} Q &= 0 \\ \nabla^C_{\partial_s} Q &= 0 \\ \nabla^C_{\dot{\gamma}} Q &= -2e^{2s} H_t^{-1}(\dot{\gamma}) \end{aligned}$$

Together with (43) this yields

$$\nabla^C_{\dot{\delta}} U = \dot{a}P + \dot{b}Q - 2be^{2s}H_t^{-1}(\dot{\gamma}) - 2\dot{s}H_t^{-1}(Z(r)) - 2e^{-2s}g_F(\dot{\gamma}, Z(r))P + H_t^{-1}(\nabla^F_{\dot{\gamma}} Z(r))$$

Since P, Q and $H_t^{-1}TF$ are independent, it follows that U is parallel along δ if and only if

$$\begin{aligned} \dot{b}(r) &= 0 \\ \dot{a}(r) &= 2e^{-2s} g_F(\dot{\gamma}, Z(r)) \\ \nabla^F_{\dot{\gamma}} Z(r) &= 2be^{2s}\dot{\gamma} + 2\dot{s}Z(r) \end{aligned}$$

If we choose the initial conditions $a(0) = b(0) = 0$ and set $Y(r) = e^{-2s}Z(r)$ we obtain, that U is parallel if and only if

$$\begin{aligned} b(r) &= 0 \\ \dot{a}(r) &= 2g_F(\dot{\gamma}, Y(r)) \\ \nabla^F_{\dot{\gamma}} Y(r) &= 0 \end{aligned}$$

This proves the Proposition. □

Theorem 3 *Let $Hol_{\hat{x}}(C, g_C)$ be the holonomy group of the cylinder $C = C[F; H]$ with respect to the point $\hat{x} \in C$, where $\hat{x} = (0, 0, x)$.*

1. *If (F, g_F) contains a flat factor (F_0, g_{F_0}), then $C[F; H]$ is decomposable.*
2. *If (F, g_F) splits (locally) into a Riemannian product of irreducible non-flat factors, then $C[F; H]$ is weakly irreducible and the connected component of its holonomy group is given by*

$$Hol^0_{\hat{x}}(C, g_C) = \left(H^{-1} \circ Hol^0_x(F, g_F)) \circ H\right) \ltimes \mathbb{R}^{dim(F)}.$$

Proof. Let us fix an ON-basis $(v_1, \ldots, v_{n-1})$ in T_xF with respect to g_F. Now, choose a closed curve $\delta(r) = (t(r), s(r), \gamma(r))$ in C with $\delta(0) = \delta(1) = (0, 0, x)$. Then for the parallel displacements hold according to Proposition 14

$$\tau^C_\delta(AZ) = a_{\gamma,Z}(1)P + A(\tau^F_\gamma(Z)) = \tau^C_\gamma(AZ)$$

where $a_{\gamma,Z}$ is determined by $\dot{a}_{\gamma,Z}(r) = 2g_F(\tau^F_{\gamma|_{[0,r]}} Z, \dot\gamma(r))$ and $a_{\gamma,Z}(0) = 0$. In particular, with respect to the basis $(P, Av_1, \dots, Av_{n-1}, Q)$ of $T_{\hat x}C$ we have

$$\tau^C_\delta = \tau^C_\gamma = \begin{pmatrix} 1 & a_\gamma & * \\ 0 & A\circ\tau^F_\gamma\circ A^{-1} & * \\ 0 & 0 & 1 \end{pmatrix} \tag{44}$$

where $a_\gamma = (a_{\gamma,v_1}, \dots, a_{\gamma,v_{n-1}}))$. (Note, that the two * in the matrix are uniquely determined by the other entries since τ^C_δ is in $SO(T_{\hat x}C, g_C)$). This implies that

$$Hol_{\hat x}(C, g_C) \subset \Big(H^{-1}\circ Hol_x(F, g_F)\circ H\Big) \ltimes \mathbb{R}^{dim(F)}.$$

Now, let us suppose, that $(F, g_F) = (F_0, g_{F_0}) \times (F_1, g_{F_1})$ is a Riemannian product of a flat factor (F_0, g_{F_0}) and another factor (F_1, g_{F_1}). Since (F_0, g_{F_0}) is flat, we have for the parallel displacement along a curve $\gamma = (\gamma_0, \gamma_1)$ with $\gamma(0) = \gamma(1) = x$

$$\tau^F_\gamma = \begin{pmatrix} \tau^{F_0}_{\gamma_0} & 0 \\ 0 & \tau^{F_1}_{\gamma_1} \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & \tau^{F_1}_{\gamma_1} \end{pmatrix}$$

Furthermore, let $(v_1 = \partial_{x_1}, \dots, v_k = \partial_{x_k})$ be a parallel coordinate basis of an Euclidean chart in F_0 and $\dot\gamma_0 = \sum_{j=1}^k \dot x_j v_j$. Then

$$\dot a_{\gamma,v_i}(r) = 2g_{F_0}(v_i, \dot\gamma_0) = 2\dot x_i(r)$$

implies $a_{\gamma,v_i}(r) = 2x_i(r) + c_i$ and from the initial condition follows $c_i = -2x_i(0)$. Hence $a_{\gamma,v_i}(1) = 2x_i(1) - 2x_i(0) = 0$ since γ is closed. Then using the above formula for the parallel displacement we obtain

$$\tau^C_\delta = \tau^C_\gamma = \begin{pmatrix} 1 & 0 & a_{\gamma_1}(1) & * \\ 0 & Id_{ATF_0} & 0 & * \\ 0 & 0 & A\circ\tau^{F_1}_{\gamma_1}\circ A^{-1} & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This shows that the non-degenerate subspace $AT_xF_0 \subset T_{(0,0,x)}C$ is holonomy-invariant. Hence in that case, the cylinder is decomposable.

Now, let $(F, g_F) = (F_1, g_{F_1}) \times \cdots \times (F_k, g_{F_k})$ be a product of irreducible Riemannian manifolds of dimension ≥ 2 and $x = (x_1, \dots, x_k)$. Let us denote by

$$\mathfrak{h} := \mathfrak{hol}_{\hat x}(C, g_C) \subset \mathfrak{so}(T_{\hat x}C, g_C)_P = \mathbb{R}P\wedge AT_xF + \mathfrak{so}(AT_xF, g_0) + \mathbb{R}P\wedge Q$$

the holonomy algebra of (C, g_C) in the point $\hat x$, where $\mathfrak{so}(T_{\hat x}C, g_C)_P$ is the Lie algebra of the stabilizer of P in $SO_0(T_{\hat x}C, g_C)$. Furthermore, let $\mathfrak{m}_i$ be the image of the projection of $\mathfrak{h}$ onto $(\mathbb{R}P\wedge AT_{x_i}F_i)$ and $\mathfrak{h}_i$ be the image of the projection of $\mathfrak{h}$ onto $\mathfrak{so}(AT_{x_i}F_i)$. From (44) we know, that $\mathfrak{h}_i = A\circ\mathfrak{hol}_{x_i}(F_i, g_{F_i})\circ A^{-1}$. Since (F, g_F) is supposed to have parallel spinors, (F_i, g_{F_i}) has parallel spinors as well. Therefore the algebras $\mathfrak{h}_i$, being $\mathfrak{su}(m), \mathfrak{sp}(m), \mathfrak{g}_2$ or $\mathfrak{spin}(7)$ each, have no center. It follows from the classification of weakly irreducible subalgebras of $\mathfrak{so}(1,n)_P$ (cf. [7] or [11]) that there is no coupling between the $\mathfrak{m}_i$ and the $\mathfrak{h}_i$-part, since such couplings can only appear, if one of the Lie algebras $\mathfrak{h}_i$ has a center. Hence

$$\mathfrak{h} = \begin{pmatrix} 0 & \mathfrak{m}_1 & \mathfrak{m}_2 & \cdots & \mathfrak{m}_k & 0 \\ 0 & \mathfrak{h}_1 & 0 & \cdots & 0 & * \\ 0 & 0 & \mathfrak{h}_2 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathfrak{h}_k & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \subset \mathfrak{so}(T_{\hat{x}}C, g_C)_P \tag{45}$$

Now, we will show that $\mathfrak{m}_i \neq 0$. Note, that by the Ambrose-Singer-Theorem, $\mathfrak{h}$ is spanned by all elements of the form

$$(\tau_\delta^C)^{-1} \circ R^C_{\delta(1)}(X,Y) \circ \tau_\delta^C$$

where $\delta : [0,1] \longrightarrow C$ runs over all curves in C starting in $\hat{x}$ and X and Y over all vectors in $T_{\delta(1)}C$. Let us consider the special curve $\delta(r) = (0,0,\gamma(r))$ where $\gamma : [0,1] \longrightarrow F$ is a geodesic in (F, g_F) starting in x . Then for the parallel displacement of a vector $Z \in T_xF$ hold by Proposition 14

$$\tau_\delta^C(AZ) = a_{\gamma,Z}(1)P + A(\tau_\gamma^F Z).$$

From Proposition 12 follows for vectors $X, Y \in TF$ (recall $P \perp H^{-1}TF$):

$$R^C(X,Y)P = 0 \qquad \text{and} \qquad R^C(X,Y)H^{-1}|_{TF} = H^{-1}R^F(X,Y)|_{TF}.$$

Since $A = H^{-1}$ this implies

$$R^C_{\delta(1)}(X,Y)\tau_\delta^C(AZ) = AR^F_{\delta(1)}(X,Y)\tau_\gamma^F(Z).$$

Let us denote by γ^- the inverse curve $\gamma^-(r) = \gamma(1-r)$. Then with $\hat{Z} := R^F_{\delta(1)}(X,Y)\tau^F_\gamma(Z)$ we obtain (recalling $R(X,Y)P = 0$)

$$(\tau_\delta^C)^{-1} \circ R^C_{\delta(1)}(X,Y) \circ \tau_\delta^C(AZ) = a_{\gamma^-,\hat{Z}}P + A\tau^F_{\gamma^-}R^F_{\gamma(1)}(X,Y)\tau^F_\gamma(Z)$$

where $a_{\gamma^-,\hat{Z}}$ is given by the differential equation

$$\dot{a}_{\gamma^-,\hat{Z}}(r) = 2g_F\Big(\dot{\gamma}^-(r), \tau^F_{\gamma^-|_{[0,r]}}R^F_{\gamma(1)}(X,Y)\tau^F_\gamma(Z)\Big) \tag{$*$}$$

with the initial condition $a_{\gamma^-,\hat{Z}}(0) = 0$. Since γ^- is a geodesic in F, the function $g_F\Big(\dot{\gamma}^-(r), \tau^F_{\gamma^-|_{[0,r]}}R^F_{\gamma(1)}(X,Y)\tau^F_\gamma(Z)\Big)$ is constant, hence the solution of the initial value problem (*) is

$$\begin{aligned} a_{\gamma^-,\hat{Z}}(r) &= 2g_F(\dot{\gamma}^-(0), R^F_{\gamma(1)}(X,Y)\tau^F_\gamma(Z)) \cdot r \\ &= -2g_F(\dot{\gamma}(1), R^F_{\gamma(1)}(X,Y)\tau^F_\gamma(Z)) \cdot r \\ &= 2g_F(R^F_{\gamma(1)}(X,Y)\dot{\gamma}(1), \tau^F_\gamma(Z)) \cdot r \end{aligned}$$

Now, assume that $\mathfrak{m}_i = 0$. Then $R^{F_i}_{\gamma_i(1)}(X,Y)(\dot{\gamma}_i(1))$ vanish for all vectors $X, Y \in T_{\gamma_i(1)}F_i$. In particular, this implies that the sectional curvature $K^{F_i}_E(\gamma_i(1))$ of F_i in the

point $\gamma_i(1)$ in direction of any 2-dimensional subspace $E = span(\tau^{F_i}_{\gamma_i}(v), \tau^{F_i}_{\gamma_i}(\dot\gamma_i(0))$ with $v \in T_{x_i}F_i$ vanishes. The same argument applies to any point $\gamma(r)$ of $\gamma((0,1])$, hence we obtain taking the limit $r \to 0$ that $K^{F_i}_{span(v,\dot\gamma_i(0))}(x_i) = 0$. If we take all geodesics γ_i starting from x_i it follows that all sectional curvatures of F_i in the point x_i vanish. Now, since the holonomy groups of different points are conjugated, with $\mathfrak{m}_i$ the projection of $\mathfrak{hol}_{\hat y}(C, g_C)$ onto the part $\mathbb{R}P \wedge AT_{y_i}F_i \subset \mathfrak{so}(T_{\hat y}C, g_C)_P$ for any other point $y \in F$ vanishes too. Then applying the same argument to y we obtain, that the sectional curvature of F_i vanishes everywhere, hence F_i has to be flat, which is a contradiction since (F_i, g_{F_i}) is irreducible and of dimension ≥ 2. Hence we have proven that the projection $\mathfrak{m}_i \neq 0$ for all components F_i of F.
It remains to prove that $\mathfrak{m}_i = AT_{x_i}F_i$ for all i. Formula (45) shows that $[\mathfrak{h}_i, P \wedge \mathfrak{m}_i] \subset P \wedge \mathfrak{h}_i(\mathfrak{m}_i)$ hence, $\mathfrak{h}_i(\mathfrak{m}_i) \subset \mathfrak{m}_i$. But by assumption, $\mathfrak{h}_i$ acts irreducible on $AT_{x_i}F_i$, hence $\mathfrak{m}_i = AT_{x_i}F_i$. This completes the proof of the second statement. □

Corollary 3 *Let an indecomposable, non-irreducible Lorentzian holonomy representation $\mathfrak{R}$ with a fixed spinor be given. Then there is a bbc manifold with holonomy representation $\mathfrak{R}$.*

Proof. First pick a complete manifold (F, g_F) having the screen bundle holonomy (cf. [17]) of $\mathfrak{R}$ as its holonomy representation. Then from any (possibly trivial) Codazzi tensor on (F, g_F) with norm uniformally bounded from below one can construct a Codazzi tensor H on the warped product over (F, g_F) with positive eigenvalues uniformally bounded away from zero. Now $C[F; H]$ is a bbc manifold with holonomy representation $\mathfrak{R}$. □

4 Examples

Our construction of Riemannian manifolds with Codazzi spinors and of Lorentzian manifolds with special holonomy (cf. Theorem 1 and Theorem 3) is based on the existence of Codazzi tensors on Riemannian manifolds with parallel spinors. Let us discuss some examples for that.

Example 1
On the flat space $\mathbb{R}^n$ the endomorphism

$$T_h^{\mathbb{R}^k}(X) = \nabla_X^{\mathbb{R}^k}(grad(h)) = X(\partial_1 h, \ldots, \partial_k h))$$

is a Codazzi tensor for any function h on $\mathbb{R}^k$ and every Codazzi tensor is of this form (cf. [10]). Proposition 12 shows, that the cylinder $C[F; H]$ is flat for any Codazzi tensor H on the warped product that is constructed out of T (cf. Corollary 2).

Example 2
Let (F_1, g_{F_1}) be a complete simply connected irreducible Riemannian spin manifold with parallel spinors and (F, g_F) its Riemannian product with a flat $\mathbb{R}^k$. Then (F, g_F) is complete and has parallel spinors. Let H be a Codazzi tensor on the warped product $\mathbb{R} \times_{e^{-2s}} F$ constructed out of the Codazzi tensor $\lambda Id_{F_1} + T_h^{\mathbb{R}^k}$ of F, where $T_h^{\mathbb{R}^k}$ is taken

from example 1. Then the cylinder $C(F;H)$ is globally hyperbolic and decomposable with special holonomy

$$Hol(F_1, g_{F_1}) \ltimes \mathbb{R}^{dimF_1}.$$

Example 3
Let us consider the metric cone

$$(F^{n-1}, g_F) := (\mathbb{R}^+ \times N, dr^2 + r^2 g_N)$$

where (N, g_N) is simply connected and a Riemannian Einstein-Sasaki manifold, a nearly Kähler manifold, a 3-Sasakian manifold or a 7-dimensional manifold with vector product. Then (F, g_F) is irreducible and has parallel spinors (but fails to be complete). Furthermore, $T := \nabla^F \partial_r$ is a Codazzi tensor on (F, g_F), since ∂_r lies in the kernel of the curvature endomorphism. Theorem 3 shows, that the cylinder $C[F;H]$, where H is constructed out of T, has special holonomy

$$Hol(C, g_C) \simeq G \ltimes \mathbb{R}^{n-1}$$

where

$$G = \begin{cases} SU((n-1)/2) & \text{if} \quad N \text{ is Einstein-Sasaki} \\ Sp((n-1)/4) & \text{if} \quad N \text{ is 3-Sasakian} \\ G_2 & \text{if} \quad N \text{ is nearly Kähler} \\ Spin(7) & \text{if} \quad N \text{ 7-dimensional with vector product} \end{cases}$$

Example 4
Let $(F, g_F) = (F_1, g_{F_1}) \times \cdots \times \ldots (F_k, g_{F_k})$ be a Riemannian product of simply connected complete irreducible Riemannian manifolds with parallel spinors. Let T be the Codazzi tensor $T = \lambda_1 \mathbf{1}_{F_1} + \ldots \lambda_k \mathbf{1}_{F_k}$ and H constructed out of T as mentioned in Corollary 2. Then $C[F;H]$ is globally hyperbolic, weakly irreducible and the holonomy group is isomorphic to

$$(Hol(F_1, g_{F_1}) \times \cdots \times Hol(F_k, g_{F_k})) \ltimes \mathbb{R}^{dimF}.$$

Example 5. Eguchi-Hansen space
We will show that there is no nontrivial Codazzi tensor on the Eguchi-Hansen space, which is an example of a complete, irreducible Riemannian 4-manifold with holonomy $SU(2)$, hence with 2 linearly independent parallel spinors.
The Eguchi-Hansen space $\overline{EH}$ is the manifold $T\mathbb{S}^2$ equipped with a metric which is ALE (asymptotically locally Euclidean). More exactly, there is a compact set whose complement converges with the metric distance to $\mathbb{R}^4/\mathbb{Z}_2$ in a certain way (cf. [14]). The complement of the zero section of $T\mathbb{S}^2$ is isometric to $((a, \infty) \times \mathbb{S}^3)/\{\pm 1\}$ with the metric

$$h_a = (1 - (\frac{a}{r})^4)^{-1} dr^2 + r^2((\sigma^x)^2 + (\sigma^y)^2) + r^2(1 - (\frac{a}{r})^4)(\sigma^z)^2,$$

where r is the parameter of (a, ∞) and $(\sigma^x, \sigma^y, \sigma^z)$ is the standard basis of left-invariant 1-forms on $\mathbb{S}^3$, i.e. $d\sigma^x = 2\sigma^y \wedge \sigma^z$ (and cyclic permutations). (Note that the metric is in fact invariant under the multiplication by -1 in $\mathbb{R}^4$!). We denote this complement of the zero section by EH as, vice versa, $\overline{EH}$ is the completion of EH by gluing in an $\mathbb{S}^2$ at $r \to a$ (cf. [22], [9]). One can extend h_a to a complete metric on $\overline{EH}$ if $a > 0$.

Theorem 4 *Let W be a Codazzi tensor on EH. Then W is a constant multiple of the identity. Moreover, on the warped products $\mathbb{R} \times_{e^{ks}} EH$, all Codazzi tensors are simple.*

Proof. Let $(\sigma_x, \sigma_y, \sigma_z)$ be the basis of left-invariant vector fields on $\mathbb{S}^3$ dual to $(\sigma^x, \sigma^y, \sigma^z)$. Then $[\sigma_x, \sigma_y] = -2\sigma_z$ (and cyclic permutations). Written in the standard spherical basis $(r\partial_r, \sigma_x, \sigma_y, \sigma_z)$, the metric becomes

$$h_a = r^2 \begin{pmatrix} (1-(\frac{a}{r})^4)^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (1-(\frac{a}{r})^4) \end{pmatrix}$$

The transformation matrix between $(r\partial_r, \sigma_x, \sigma_y, \sigma_z)$ and the canonical basis of the Euklidian coordinates on $\mathbb{C}^2 = \mathbb{R}^4$ at a point (x_1, x_2, x_3, x_4) is

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & x_4 & -x_3 \\ x_3 & -x_4 & -x_1 & x_2 \\ x_4 & x_3 & -x_2 & -x_1 \end{pmatrix}$$

For short, we denote by f the function

$$f(r) = \left(1 - \left(\frac{a}{r}\right)^4\right)^{1/2}.$$

Note, that for this function

$$f' + r^{-1}f = 2r^{-1}f^{-1} - r^{-1}f =: \gamma(r).$$

Then

$$\sigma^0 = f^{-1}dr\ ,\ \ \sigma^1 = r\sigma^x\ ,\ \ \sigma^2 = r\sigma^y\ ,\ \ \sigma^3 = rf\sigma^z$$

is an orthogonal coframe for the metric h_a and

$$e_0 = f\partial_r\ ,\ \ e_1 = r^{-1}\sigma_x\ ,\ \ e_2 = r^{-1}\sigma_y\ ,\ \ e_3 = r^{-1}f^{-1}\sigma_z$$

is the dual orthonormal basis for h_a. For the commutators we obtain

$$\begin{aligned} [e_0, e_1] &= -r^{-1}f\, e_1 \\ [e_0, e_2] &= -r^{-1}f\, e_2 \\ [e_0, e_3] &= -(f' + r^{-1}f)\, e_3 \;=\; -\gamma\, e_3 \\ [e_1, e_2] &= -2r^{-1}f\, e_3 \\ [e_1, e_3] &= +2r^{-1}f^{-1}\, e_2 \\ [e_2, e_3] &= -2r^{-1}f^{-1}\, e_1 \end{aligned}$$

Then using the Koszul formula

$$2\langle \nabla_{e_i} e_j, e_k \rangle = -\langle e_i, [e_j, e_k]\rangle - \langle e_j, [e_i, e_k]\rangle + \langle e_k, [e_i, e_j]\rangle$$

we obtain for the Levi-Civita connection of $h_a = \langle \cdot, \cdot \rangle$

$$\nabla_{e_0} e_k = 0 \qquad k = 0,1,2,3$$

$$\begin{aligned}
\nabla_{e_1} e_0 &= +r^{-1} f e_1 & \nabla_{e_2} e_0 &= +r^{-1} f e_2 \\
\nabla_{e_1} e_1 &= -r^{-1} f e_0 & \nabla_{e_2} e_1 &= +r^{-1} f e_3 \\
\nabla_{e_1} e_2 &= -r^{-1} f e_3 & \nabla_{e_2} e_2 &= -r^{-1} f e_0 \\
\nabla_{e_1} e_3 &= +r^{-1} f e_2 & \nabla_{e_2} e_3 &= -r^{-1} f e_1
\end{aligned}$$

$$\begin{aligned}
\nabla_{e_3} e_0 &= +(r^{-1} f + f')\, e_3 \;=\; +\gamma\, e_3 \\
\nabla_{e_3} e_1 &= +(r^{-1} f - 2r^{-1} f^{-1})\, e_2 \;= -\gamma\, e_2 \\
\nabla_{e_3} e_2 &= -(r^{-1} f - 2r^{-1} f^{-1})\, e_1 \;= +\gamma\, e_1 \\
\nabla_{e_3} e_3 &= -(r^{-1} f + f')\, e_0 \;=\; -\gamma\, e_0
\end{aligned}$$

We consider W in the basis e_a mentionned above. The first observation is that w.r.o.g the entries of the associated matrix only depend on r, not on the spherical variables. That can be seen as follows: Assume that there is a Codazzi tensor field W whose entries might depend on all variables. Then consider

$$\tilde{W} := \int_{\mathbb{S}^3} (\mathbf{1} \times L_q)^* W dq$$

where L_q is the left action of the sphere on itself. Remind that the Berger metrics are left-invariant, thus the diffeomorphisms $\mathbf{1} \times L_q$ are isometries of EH. Therefore the endomorphisms $(\mathbf{1} \times L_q)^* W$ are Codazzi. As the Codazzi equation is linear in the tensor field, it commutes with the integral, and $\tilde{W}$ is Codazzi. As the frame e_a is left-invariant, the entries of $\tilde{W}$ cannot depend on the spherical coordinates any more but only on r. Up to know we know that w.r.o.g. we have a tensor field of the form

$$W := \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}$$

with $A, B, C, D, E, F, G, H, I, J$ real functions depending on r. The second observation is that, for U being the endomorphism which exchanges e_1 and e_2 and is the identity on the orthogonal complement of $\{e_1, e_2\}$, if W is a Codazzi tensor, the endomorphism $U \circ W \circ U$ is a Codazzi tensor as well. Thus the endomorphism fields $W \pm U \circ W \circ U$ are Codazzi tensors as well which are symmetric resp. antisymmetric under the conjugation by U. We will show that the U-antisymmetric part vanishes necessarily while the U-symmetric part has to be a constant multiple of the identity.
The U-antisymmetric part has $A = D = F = J = 0$ and $B = -C$, $E = -H$, $G = -I$. Therefore it looks like

$$W := \begin{pmatrix} 0 & B & -B & 0 \\ B & E & 0 & G \\ -B & 0 & -E & -G \\ 0 & G & -G & 0 \end{pmatrix}$$

Now we consider the Codazzi equation for the vectors e_1, e_2:

$$
\begin{aligned}
0 &= \nabla_{e_1}(W(e_2)) - \nabla_{e_2}(W(e_1)) - W([e_1, e_2]) \\
&= \nabla_{e_1}(-Be_0 - Ee_2 - Ge_3) - \nabla_{e_2}(Be_0 + Ee_1 + Ge_3) - W(-2\frac{f}{r}e_3) \\
&= \frac{f}{r}((-Be_1 + Ee_3 - Ge_2) - (Be_2 + Ee_3 - Ge_1) + 2(Ge_1 - Ge_2)) \\
&= \frac{f}{r}\big((-B + 3G)e_1 + (-B - 3G)e_2\big)
\end{aligned}
$$

which implies $B = G = 0$. Thus W looks like

$$
W := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

The same procedure applied to the pair e_1, e_3 leads to

$$
\begin{aligned}
0 &= \nabla_{e_1}(W(e_3)) - \nabla_{e_3}(W(e_1)) - W([e_1, e_3]) \\
&= -E\nabla_{e_3}e_1 - W(2r^{-1}f^{-1}e_2) \\
&= E\gamma e_2 + E2r^{-1}f^{-1}e_2
\end{aligned}
$$

which implies $E = 0$ as $\gamma = 2r^{-1}f^{-1} - r^{-1}f$.
On the other hand, the U-symmetric part has $C = B, H = E, G = I$, i.e. that

$$
W := \begin{pmatrix} A & B & B & D \\ B & E & F & G \\ B & F & E & G \\ D & G & G & J \end{pmatrix}
$$

Now consider the Codazzi equation for the vectors e_1, e_2:

$$
\begin{aligned}
0 &= \nabla_{e_1}(W(e_2)) - \nabla_{e_2}(W(e_1)) - W([e_1, e_2]) \\
&= \nabla_{e_1}(Be_0 + Fe_1 + Ke_2 + Ge_3) - \nabla_{e_2}(Be_0 + Ke_1 + Fe_2 + Ge_3) - W(-2\frac{f}{r}e_3) \\
&= \frac{f}{r}((Be_1 - Fe_0 - Ke_3 + Ge_2) - (Be_2 + Ke_3 - Fe_0 - Ge_1) + 2(De_0 + Ge_1 + Ge_2 + Je_3)) \\
&= 2De_0 + (B + 3G)e_1 + (-B + 3G)e_2 + (2J - 2K)e_3
\end{aligned}
$$

which implies $D = B = G = 0$. Therefore W has necessarily the form

$$
W := \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & K & F & 0 \\ 0 & F & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix}
$$

But then the Codazzi equation applied to the vectors e_1, e_3 reads

$$
\begin{aligned}
0 &= \nabla_{e_1}(W(e_3)) - \nabla_{e_3}(W(e_1)) - W([e_1, e_3]) \\
&= K\nabla_{e_1}e_3 - K\nabla_{e_3}e_1 - F\nabla_{e_3}e_2 - 2r^{-1}f^{-1}(Fe_1 + Ke_2) \\
&= F(-\gamma - 2r^{-1}f^{-1})e_1
\end{aligned}
$$

which can never be satisfied unless $F = 0$ as $\gamma = 2r^{-1}f^{-1} - r^{-1}f$. Thus W must have the form

$$W := \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix}$$

Finally, we consider the vectors e_0, e_1 and the vectors e_0, e_3. The Codazzi equation of the first pair is equivalent to the condition $e_0(K) = \frac{f}{r}(A - K)$, the Codazzi equation of the second pair is equivalent to $e_0(K) = \gamma(A - K)$. Combined this yields $(\frac{f}{r} - \gamma)(A - K) = 0$, thus $A = K$ and $e_0(K) = 0$, thus we have shown that the U-symmetric part is a constant multiple of the identity on EH which completes the proof of the first part of the Theorem. The second part is proven analogously: First we assume the existence of a nonzero homothetic vector field on EH, i.e. of a vector field V with $\nabla.V = c \cdot \mathbf{1}$. By integrating over $\mathbb{S}^3$ we show that all coefficients of V w.r.t. the left-invariant basis above cannot depend on the spherical variables but only on r. Thus $V = Ae_0 + Be_1 + Ce_2 + De_3$ with A, B, C, D real functions depending only on r. Then we compute

$$ce_1 = \nabla_{e_1} V = A\nabla_{e_1} e_0 + B\nabla_{e_1} e_1 + C\nabla_{e_1} e_2 + D\nabla_{e_1} e_3$$

and as only $\nabla_{e_1} e_0$ is collinear to e_1, we conclude that $B = C = D = 0$ and $c = \frac{f}{r}A$. But the same procedure for e_3 instead of e_1 gives $c = \gamma A$. Thus both equations together can be satisfied only for $A = 0$ as $\gamma = 2r^{-1}f^{-1} - r^{-1}f$. Thus there is no nonzero vector field V on EH with $\nabla.V = c \cdot \mathbf{1}$. Therefore Corollary 1 implies that on $\mathbb{R} \times_{e^{ks}} EH$ there are no non-simple Codazzi tensors. □

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