Compact Lorentzian holonomy *

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Abstract

We consider (compact or noncompact) Lorentzian manifolds whose holonomy group has compact closure. This property is equivalent to admitting a parallel timelike vector field. We give some applications and derive some properties of the space of all such metrics on a given manifold.

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1 Introduction

It is well known that a structure group reduction of the frame bundle encodes the existence of a geometric structure on the manifold. If, moreover, it contains the holonomy group of a given connection ∇, the geometric structure is ∇-parallel, [20, Propositions 5.6 and 7.4]. The most familiar example is the existence of a semi-Riemannian metric which is equivalent to a reduction of the structure group to $O_\nu(n)$. Metricity of the Levi-Civita connection implies that its holonomy group is contained in $O_\nu(n)$. Another classical example is a $2n$-dimensional Kähler manifold. It has holonomy group contained in $U(n)$. In fact, $U(n) = GL(n, \mathbb{C}) \cap Sp(n, \mathbb{R}) \cap O(2n)$, and this means that the manifold has a complex structure and a symplectic structure which are parallel and adapted to a Riemannian metric.

In (oriented) Riemannian geometry, the generic holonomy is the (special) orthogonal group, so noncompact (i.e., non-closed) holonomy implies the presence of a parallel geometric structure. Simply connected Riemannian manifolds have compact holonomy group because it coincides with its restricted holonomy group, which is well known to be compact, [7]. On the other hand, the question

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of the existence of a compact Riemannian manifold with noncompact holonomy was solved in [26] where the author showed the existence of such manifolds and studied their structures. In fact, a compact Riemannian manifold with noncompact holonomy has a finite cover that is the total space of a torus bundle over a compact manifold, and its dimension is greater or equal than 5.

Lorentzian holonomy groups have attracted much attention in the last years — for overview articles on this topic we refer to [3], [12]. [13]. Now, the situation in Lorentzian manifolds is similar but slightly different because the generic holonomy is the Lorentz group which is noncompact. It is natural to ask the analogous question: can we describe the Lorentzian manifolds which have compact holonomy?

Noncompactness of the holonomy group is responsible for noncompleteness in some compact Lorentzian manifods, as in the Clifton-Pohl torus. The relationship between holonomy and completeness is in general not well known, see e.g. [21] were the authors study the case of compact pp-waves. It is related to undesirable identifications of singular points in b-singularity theory, [25]. In fact, in [1] it was shown that in the four dimensional Friedmann model of the Universe, which have noncompact holonomy, big bang and big crunch are the same point in the b-boundary. On the other hand, compactness of the holonomy group has been used in [14] to define so-called Cauchy singular boundaries in space-times. Later, one of the authors (M. G.), using the fundamental observation that both a Lorentzian metric g and its flip Riemannian metric around a parallel timelike vector field induce the same Levi-Civita connection, proved that compactness of the holonomy group implies that the Cauchy singular boundary of the manifold is homeomorphic to its b-boundary, [17].

In this article we identify Lorentzian manifolds (compact or not) whose holonomy groups have compact closure (Theorem 2), and draw some consequences. For example, we characterize the case $Hol \subset SO(m-k)$, $k \in \{1, ..., m\}$ (Theorem 7) and use it to show that in the category of complete semi-Riemannian manifolds, if they can be decomposed in a direct product in a weak sense, then it is generically unique between direct product decomposition (weak or not), and this property fails only in 2-dimensional Minkowski or Euclidean spaces (Theorem 12). This is a rigidity type result that cannot be achieved directly from the uniqueness of the De Rham-Wu theorem.

Motivated by conceptual questions around the Lorentzian Einstein equation and led by the characterization given in Theorem 2, we then initiate a study on various topologies on the set $G(M)$ of globally hyperbolic Lorentzian metrics on a manifold M, in topologies induced by usual topologies on the space $\mathrm{Bil}(M)$ of bilinear forms on M. We find that the closure of $K(M)$, the subset of metrics with precompact holonomy, in the compact-open topology, consists of metrics with parallel causal vector fields (see Theorem 13). Moreover, as each connected component of $G(M)$ with the C^0-fine topology is formed by metrics with diffeomorphic Cauchy surfaces, we can show that $G(\mathbb{R}^n)$ with $n \geq 4$ has uncountably many componentes each of which intersects $K(M)$. Finally, if $g \in K(M)$ is timelike complete, then any other metric in the same C^1-fine path connected component of $K(M)$ is isometric to g, Corollaries 16 and 18.

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2 Compact holonomy

We assume all manifolds to be connected. Let (M, g) be a semi-Riemannian manifold with signature ν. We denote $Hol(M, g)$ and $Hol^0(M, g)$ its holonomy group and its restricted holonomy group, respectively. We drop M if no confusion is possible. $Hol_p(M, g)$ and $Hol^0_p(M, g)$ will refer to the resp. holonomy group at a given point (whereas the corresponding objects without specifying a point are, strictly speaking, only *equivalence classes of subgroups of* $Gl(n)$ *or of representations*). Observe that whereas in the Riemannian case, the reduced holonomy is always a closed subgroup of $SO(n)$ [7], there are examples of simply connected Lorentzian manifolds with non-closed holonomy group, [5].

The following theorem identifies Lorentzian manifolds with holonomy contained in a compact group. We use the following well known lemma

Lemma 1 *The map*

$$\pi_1(M, p) \quad \stackrel{j}{\longrightarrow} \quad Hol(M, g)/Hol^0(M, g)$$

given by $j([\gamma]) = [p_\gamma]$ *(where* p_γ *is the parallel transport along* γ*) is a surjective group morphism.*

The following theorem could be derived as a consequence of the well know fundamental principle (see [6, 10.19]), combined with the classification of subgroups of the Lorentz group, and the above lemma. However, for later use, we prefer to give a proof using the Haar measure.

Theorem 2 *Let* (M, g) *be a time oriented Lorentzian manifold.*

1. *The holonomy group is relatively compact (and thereby contained in a compact group) if and only if it admits a timelike parallel vector field.*

2. *If* (M, g) *admits a timelike parallel vector field and* $\pi_1(M, p)$ *is finite, then its holonomy group is compact.*

Remark 3 *Of course, a parallel timelike vector field induces a local product splitting of the manifold. But a local or global splitting does not suffice to imply that the holonomy is contained in a compact group, see Remark (8) below for a counterexample.*

It is also not true that the holonomy is always compact if there is a parallel timelike vector field, see Remark (5).

Remark 4 *Note that the proof below does not work in higher signature because the stabilizer of a nontrivial vector is not compact, so it is not possible to distinguish a non trivial parallel vector field.*

Remark 5 *The hypothesis on the finiteness of $\pi_1(M,p)$ is necessary to ensure compact holonomy. A counterexample are the direct products $(\mathbb{S}^1, -dt^2) \times (T, g_0)$ and $(\mathbb{R}^1, -dt^2) \times (T, g_0)$ whose holonomy groups equal the one of (T, g_0) and we can choose it with noncompact holonomy, as soon as $|\pi_1(T)| = \infty$.*

Proof: As $Hol(M,g)$ is contained in a compact subset, its closure C, which is a subgroup as well, is compact, and thus carries a bi-invariant Haar measure μ. Now let a point $p \in M$ be given. We want to construct a timelike vector $v \in T_pM$ invariant under Hol_p. To that purpose, choose a future timelike vector $v_0 \in T_pM$ at will and define $v := \int_{Hol_p} h(v_0) d\mu(h)$. The integral exists as the Haar measure of the compact group C is finite and the action is continuous. Now, given any $k \in Hol_p$, we compute

$$kv = \int_{Hol_p} k \circ h(v_0) d\mu(h) = \int_{Hol_p} h(v_0) k^* d\mu(h) = \int_{Hol_p} h(v_0) d\mu(h) = v,$$

so indeed v is invariant, and it is timelike, as the integrand consists in timelike future vectors and those form a convex set. And now, using the parallel transport P_c along a curve c, we have that $P_c(v) = P_k(v)$ if $c(0) = p = k(0)$ and $c(1) = k(1)$, because P_k is an isomorphism and $P_k^{-1} \circ P_c(v) = P_{ck^{-1}}(v) = v$ as $P_{ck^{-1}} \in Hol_p$. Thus there is a well-defined way to extend v to a parallel future timelike vector field V. The second item follows easily from the fact that Hol_0 is connected [7], from Lemma 1 and the fact that there is no finite quotient of a noncompact group by a compact one. ■

Let us show a statement adapted to a submanifold. We define, for a submanifold $S \subset M$, and a point $x \in S$

$$Hol(S, M, g) := \{P_c \ / \ c(0) = x = c(1), \ c'(s) \in TS \ \forall s \in [0,1]\}$$

where P_c is the parallel transport w.r.t. the connection of the ambient manifold M, and call $Hol(S, M, g)$ the *relative holonomy of* S. The independence on the point x is up to conjugation, just as in the case of usual (absolute) holonomy.

Theorem 6 *If S is a spacelike totally geodesic submanifold of a time-oriented Lorentzian manifold (M,g) and if $Hol(S,M,g)$ (resp. $Hol^0(S,M,g)$) is precompact, then the normal bundle of S contains a section that is invariant under $Hol(S,M,g)$ (resp. under $Hol^0(S,M,g)$).*

Proof: The hypothesis on $Hol(S,M,g)$ fixes a temporal vector field V like in the proof above, which we can assume to be future. As TS is invariant by $Hol(S,M,g)$, we know that $W := pr^g_{TS} V$ and $\tilde{V} := V - W$ are fixed by $Hol(S,M,g)$ as well. As the latter is normal, this proves the claim. ■

Theorem 7 *Let (M,g) be an oriented and time oriented m-dimensional Lorentzian manifold. Then, $Hol \subset SO(m-k)$, $k \in \{1, ..., m\}$ if and only if M admits an orthonormal system $\{V_1, V_2, ..., V_k\}$ formed by parallel vector fields, with V_1 timelike.*

Proof: It is clear that it is true for $k = 1$ after Theorem 2 and as the conjugacy class of $SO(m-1)$ is the maximal compact conjugacy class of subgroups of $SO(1, m-1)$. Suppose it is true for $k-1$. If $\{e_1, e_2, ..., e_{k-1}\}$ is an ortonormal system in T_pM with $e_i = V_i(p)$, then, Hol keeps $V_1, ..., V_{k-1}$ invariant and $SO(m-k)$ is the stabilizer of a timelike k-dimensional subspace L^k of T_pM containing the above system$\{e_1, e_2, ..., e_{k-1}\}$. We can complete this system to an orthonormal basis of L^k choosing a spacelike vector $e_k \in L^k$. This induce an Hol invariant vector field V_k showing that the theorem is true for k.

Conversely, let $U \subset T_pM$ be the subspace generated by $\{V_1, V_2, ..., V_k\}_p$. Each element $h \in Hol$ decomposes as

$$id \oplus h_2 : U \oplus U^{\perp} \to U \oplus U^{\perp}$$

where h_2 acts as an isometry on $U^{\perp}$. ∎

Remark 8 *The existence of a Lorentzian manifold with a timelike parallel vector field V and Hol noncompact is clear in the noncompact case because we turn the question into a well-known Riemannian one using the flip metric*

$$g_R(X, Y) = g(X, Y) + 2g(X, V)g(Y, V) \tag{1}$$

with $g(V, V) = -1$ (and thus $g_R(V, V) = 1$) which share the Levi-Civita connection with the Lorentzian metric g. The compact case is more involved, but it can be solved using the results in [26]. We consider three cases

- $\dim M \geq 6$, *the above example $M = \mathbb{S}^1 \times T$ with T compact and $Hol(T)$ noncompact shows that they exist, but in this case we know that* $\dim T \geq 5$.
- $\dim M \leq 4$, *the presence of a parallel timelike vector field allows us construct the Riemannian flip metric on M, and this implies that the holonomy is compact.*
- $\dim M = 5$. *We can not apply neither of the above direct arguments, but the Wilking example provides one.*

  Consider a semidirect product $S = \mathbb{R}^4 \rtimes \mathbb{R}$, and a discrete cocompact subgroup Λ which acts as deck transformation group of the covering $p : S \to \Lambda \backslash S$ by left translations. The group S admits a left invariant metric $g = \langle \cdot, \cdot \rangle \times g_1$ where $\langle \cdot, \cdot \rangle$ is the euclidean metric in $U = \mathbb{R}^2 \times \{0\}$ and g_1 is a left invariant metric on $S_1 = \{0\} \times \mathbb{R}^2 \rtimes \mathbb{R}$. The Wilking example is the quotient $\Lambda \backslash S$ with the induced metric with the choice of two parameters. It has non compact holonomy group. The left invariant vector field $V \in \mathfrak{X}(S)$ defined by $(1, 0, 0, 0, 0) \in \mathfrak{s}$, where $\mathfrak{s}$ is the Lie algebra of S, is invariant by Λ so it defines a vector field $V \in \mathfrak{X}(\Lambda \backslash S)$. It is clear that both vector fields are parallel.

  Using the flip metric in (1) we get a Lorentzian metric on a compact manifold $\Lambda \backslash S$ with V a timelike parallel vector field and non compact holonomy.

As an application of Theorem 2, we see directly that some kind of manifolds do not admit Lorentzian metrics with relatively compact holonomy, for example odd spheres (which *do* admit Lorentzian metrics because of vanishing Euler number but which are not direct products).

We compare the holonomy groups in a covering space. Let $\pi : M \longrightarrow B$ be a semi-Riemannian covering, so both M and B have the same restricted holonomy group.

Lemma 9 *Let $\pi : M \longrightarrow B$ be a semi-Riemannian covering map.*

1. *The map $\pi^{\#} : Hol(M) \longrightarrow Hol(B)$ given by $\pi^{\#}(P_\gamma) = P_{\pi\circ\gamma}$, is a Lie group monomorphism.*

2. *If $Hol(B)$ is compact, then $Hol(M)$ is also compact.*

3. *If $Hol(M)$ is compact and $\pi_1(B,p)$ is finite, then $Hol(B)$ is also compact.*

Proof: Observe that $P_\gamma P_\beta = P_{\beta\gamma}$ and $P_{\pi(\beta\gamma)} = P_{\pi(\gamma)}P_{\pi(\beta)}$ for any couple of lassos γ, β at p. On the other hand, π is a local isometry so $\pi_{*p}P_\gamma = P_{\pi(\gamma)}\pi_{*p}$. Thus if $P_\gamma = P_\beta \in Hol^M$, we have $e = P_{\beta^{-1}\gamma}$ being e the identity element, and applying π_{*_p} implies that $P_{\pi(\gamma)} = P_{\pi(\beta)}$. This shows that $\pi^{\#}$ is well defined.

1. We see that it is a morphism using $P_\gamma P_\beta = P_{\beta\gamma}$ and $P_{\pi(\beta\gamma)} = P_{\pi(\gamma)}P_{\pi(\beta)}$. To see that it is injective use $\pi_{*p}P_\gamma = P_{\pi(\gamma)}\pi_{*p}$.

2. Observe that $Hol^0(B) = Hol^0(M)$ is compact, so the connected components of $Hol(M)$ are diffeomorphic to $Hol^0(B)$ and $Hol(M)$ itself can be identified to its image by $\pi^{\#}$ in $Hol(B)$. Finally, $Hol(B)$ has a finite number of connected components because it is compact.

3. Note that the hypothesis implies $\# Hol(B)/Hol^0(B) < \infty$ by Lemma 1, and $Hol^0(B) = Hol^0(M)$ is compact, thus $Hol(B)$ is also compact.

■

Given $u, v \in T_pM$ where u is a null vector, it is defined the null sectional curvature of the degenerate plane $\pi = span\{u, v\}$ as

$$\mathcal{K}_u(\pi) = \frac{g(R_{uv}v, u)}{g(v, v)}.$$

It depends on the null vector $u \in T_pM$, but once it is fixed, it is a map on degenerate planes in T_pM containing u. If we fix a timelike vector field U, we can see $\mathcal{K}_U$ as a map on the subset of degenerate planes in the Grassmannian of planes in TM, defining $\mathcal{K}_U(\pi) := \mathcal{K}_u(\pi)$ where $u \in \pi$ is the unique null vector such that $g(u, U) = 1$. There are examples where $\mathcal{K}_U$ is in fact a map from M, that is, it does not depend on the choice of degenerate plane $\pi \subset T_pM$ but just on the point p itself. In this case we say that it is a pointwise function. It is a strong condition, in some sense similar to the well known Schur Lemma in the Riemannian case, see [19] for details. The sign of $\mathcal{K}_U$ does not depend on the chosen vector field U. In fact, if U' is another timelike vector field, and $u \in T_pM$ is the unique null vector in π with $g(u,U)=1$,

$$\mathcal{K}_U(\pi) = g(u,U')^2 \mathcal{K}_{U'}(\pi).$$

So it is reasonable to speak of positive null sectional curvature for all degenerate planes, [18].

The following result shows that null curvature can help to determine a Lorentzian holonomy.

Proposition 10 *Let (M,g) be a complete and non-compact Lorentzian manifold with $m = \dim M \geq 4$ such that the null sectional curvature is a positive pointwise function. If the holonomy group is contained in a compact group, then $Hol(M) = SO(m-1)$ or $O(m-1)$.*

Proof: A suitable finite covering $\widetilde{M}$ of M is orientable and time orientable, so Lemma 9 and Theorem 2 tell us that $\widetilde{M}$ admits a timelike parallel vector field U. In particular, we can use U as a geodesic reference frame in the sense of [15] to deduce that $\widetilde{M}$ is a direct product $\mathbb{R} \times L$ where the second factor is a quotient of the usual sphere $\mathbb{S}^{m-1}$ of constant positive curvature, see [15, Proposition 5.4]. The fact that L is a quotient of $\mathbb{S}^{m-1}$ and Lemma 9 again implies $SO(m-1) \subset Hol(\widetilde{M}) \subset Hol(M)$. By hypothesis, $Hol(M)$ is contained in a compact group, in particular in a maximal compact one, that is, in a copy of $O(m-1)$. ■

Let us consider another consequence of Theorem 2. It is a well-known result by Marsden [22] that a compact homogeneous semi-Riemannian manifold is geodesically complete (whereas the same is not true omitting the condition of homogeneity). Using a suitable covering and [24], we have

Corollary 11 *Let M be a compact manifold and let g be a Lorentzian metric on M with precompact holonomy. Then (M,g) is geodesically complete.*

This result is a particular case of a more general result valid for holonomy groups defined for an arbitrary connection, see [2].

Inspired in [16], we can show that Euclidean and Minkowski plane are the unique semi-Riemannian manifolds with the property that they admit another direct product decomposition with non degenerate properties.

A semi-Riemannian manifold is called *(locally) indecomposable* iff its holonomy group is *weakly irreducible*, i.e. iff the latter does not admit non-trivial and nondegenerate invariant subspaces by the holonomy group in any tangent space. In the Riemannian case this notion coincides with the usual notion of irreducibility. Given a manifold $M = M_1 \times M_2$, we call $M_i(p)$ the tangent space at p of the leaf of the i-th canonical foliation through $p \in M$.

Theorem 12 *Let $M = M_1 \times M_2$ be a complete semi-Riemannian direct product with M_i indecomposable. Suppose that M admits another decomposition as a direct product $M = L_1 \times L_2$ (with $L_1 \neq M_i$), and $M_i(p) \cap L_j(p)$ zero or non degenerate. Then $M = \mathbb{R}^2$, the euclidean or Minkowski plane.*

Proof: Suppose that $\dim M_1 = k$, and the signature of M_i is ν_i, such that the signature of M is $\nu = \nu_1 + \nu_2$. Let $i : O_{\nu_1}(k) \longrightarrow O_\nu(m)$ and $j : O_{\nu_2}(m-k) \longrightarrow O_\nu(m)$ be the natural inmersions $i(c) = \begin{pmatrix} c & 0 \\ 0 & I_{m-k} \end{pmatrix}$, $j(d) = \begin{pmatrix} I_k & 0 \\ 0 & d \end{pmatrix}$. We call $G = i(O_{\nu_1}(k))j(O_{\nu_2}(m-k))$. It is clear that if M is a direct product $M_1 \times M_2$, its holonomy group H is reducible to a subgroup of G, that is, $H = H_1 H_2$ with $H_1 \subset i(O_{\nu_1}(k))$ and $H_2 \subset j(O_{\nu_2}(m-k))$. Let $\pi : OM \longrightarrow M$ be the orthonormal frame bundle. Call

$$E = \{r \in OM \ / \ r : \mathbb{R}^m \longrightarrow T_{\pi r}M \ \textit{such that carries adapted}$$
$$\textit{basis of } \mathbb{R}^k \times \mathbb{R}^{m-k} \textit{ to adapted basis of } M_1 \times M_2\}.$$

With respect to the decomposition $M = L_1 \times L_2$, fixed an element $r \in E$, there exists another decomposition of $\mathbb{R}^m$ as a direct product $S_1 \times S_2$ such that r carries an adapted basis of $S_1 \times S_2$ to an adapted basis of $L_1 \times L_2$.

Both tuples of foliations in M are invariant by parallel transport, that is, the subspaces $\mathbb{R}^k$, $\mathbb{R}^{m-k}$, S_1 and S_2 of $\mathbb{R}^m$ are invariant by the holonomy group H.

Given $h \in H_1$, we can write $h = \begin{pmatrix} c & 0 \\ 0 & I \end{pmatrix}$ with $c \in O_{\nu_1}(k)$, and if we call (x_1, x_2) the components of $x = \pi(r) \in M$ in $M_1 \times M_2$ and (x_1', x_2') its components in $L_1 \times L_2$, we have the following two ways in which we can write the composition $r \circ h$

$$\begin{array}{ccccc} \mathbb{R}^k \times \mathbb{R}^{m-k} & \xrightarrow{h} & \mathbb{R}^k \times \mathbb{R}^{m-k} & \xrightarrow{r} & T_{x_1}M_1 \times T_{x_2}M_2 \\ S_1 \times S_2 & \xrightarrow{h} & S_1 \times S_2 & \xrightarrow{r} & T_{x_1'}L_1 \times T_{x_2'}L_2. \end{array}$$

Given $(u,0) \in S_1 \times \{0\}$, we have $h(u,0) \in S_1 \times \{0\}$ because S_1 is invariant by H. On the other hand if we write u with its components in the other decomposition, $u = (u_1, u_2) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$, we have $h(u) = (cu_1, u_2)$ and

$$u - h(u) = (u_1 - cu_1, 0) \in (S_1 \cap \mathbb{R}^k) \times \mathbb{R}^{m-k}.$$

By hypothesis $S_1 \cap \mathbb{R}^k$ is zero or a non degenerated subspace of $\mathbb{R}^k$ invariant by H, but the holonomy of M_1 is weakly irreducible, so it must be zero, thus $H_1 = \{1\}$. A similar argument for H_2 implies that $H = \{1\}$. Theorem 7 implies that M admits a global orthonormal basis formed by parallel vector fields $E_1, \ldots E_m$. By completeness, the universal covering $\widetilde{M}$ splits as $\mathbb{R}^m$ with a flat metric. The group of deck transformation preserves the parallel basis, otherwise H would not be trivial, thus M_i is a product of $m_i = \dim M_i$ factors each one being $\mathbb{R}$ or $\mathbb{S}^1$, but the holonomy of M_i being weakly irreducible we have $m_i = 1$, therefore $m = \dim M = 2$. The only complete flat surfaces that admits two different structures as a direct product are the euclidean and the Minkowski plane. ■

Note that in this proof, we do not suppose a priori that L_i must also be indecomposable nor M simply connected. This is of crucial importance because if we suppose them, the uniqueness of the decomposition in the Theorem of de Rham-Wu can be used to give a direct proof, [28], [29, Apendix I], [10]. In fact, M should be isometric to $\mathbb{R}^m$ with flat metric and $H = \{1\}$. So M_i is a direct product of $m_i = \dim M_i$ factors $\mathbb{R}^1$, but M_i indecomposable imposes $m_i = 1$. Thus $\dim M = 2$.

3 Topologies on the space of all metrics with precompact holonomy

Having made several statements about single metrics with precompact holonomy, let us try to explore the topology of the space of *all* time-oriented metrics on an orientable manifold M that have precompact holonomy, in analogy to the situation in positive curvature. However, it will turn out in the following that much of its topology is hidden behind the not quite accessible topology of the space of Lorentzian metrics. In the light of applications like the Einstein equation considered as a variational problem, it is outermost desirable to construct an appropriate topology on the space of Lorentzian metrics. Several topologies on that space and on related spaces have been considered. In a row of articles, Bombelli, Meyer, Noldus and Sorkin, e.g., introduced topologies on the quotient $\mathrm{Lor}(M)/\mathrm{Diff}(M)$ based on a splitting between the conformal and the volume part (for an overview, see [23]), but unfortunately, this topology is not a manifold topology in general. As we are ultimately interested in variational problems, and thus look for a manifold topology on the space $\mathrm{Lor}_+(M)$ of time-oriented Lorentzian metrics, the simplest choice is the subspace topology with respect to a topological vector space topology on the space $\mathrm{Bil}(M)$ of bilinear forms on M. Let

$$PT(M) := \{g \in \mathrm{Lor}_+(M) \ / \ \overline{Hol_g} \text{ compact}\}$$

thus, following Theorem 2, $PT(M)$ is the set of time-oriented Lorentzian metrics with a parallel timelike vector field.

We define $G(M)$ to be the space of all globally hyperbolic metrics and $C(M)$ to be the set of all *causally complete*, i.e. timelike and lightlike complete, metrics on M.

First of all we want to compare the different possible topologies on $PT(M)$ (understood as a subset of $\mathrm{Lor}(M)$). On one hand, if M is noncompact, only a topology at least as fine as the C^0-fine (Whitney) topology on $\mathrm{Bil}(M)$ ensures that $\mathrm{Lor}(M)$ is an open subset of $\mathrm{Bil}(M)$. On the other hand, as we want to be able to define parallel vector fields, all metrics should at least be C^1, and thus, the fact that we want to have a *complete* vector space topology on $Bil(M)$, recommends us to choose a topology at least as strong as the C^1-compact-open topology. First of all, for $PC(M)$ being the set of time-oriented Lorentzian metrics with a parallel causal vector field, we observe that we can control the closure of $PT(M)$ in terms of $PC(M)$:

Theorem 13 *[Closure of $PT(M)$]Let M be a manifold.*

1. *In any topology finer or equal to the C^1-compact-open topology, the closure of the set $PT(M)$ is contained in $PC(M)$.*

2. *If M is diffeomorphic to $\mathbb{R} \times S$ for some manifold S (to ensure $G(M) \neq \emptyset$), and if moreover the Euler characteristic of S vanishes (in particular if S is open), then for $E := PC(M) \cap G(M) \cap C(M)$ and $F := PT(M) \cap G(M) \cap C(M)$, there is some $e \in PC(M) \cap C(M) \cap G(M) \setminus PT(M)$ and a curve $c : [0,1] \to E$ that is smooth w.r.t. every C^k-compact open topology with $c(1) = e$ and with $c([0,1)) \subset F$.*

Remark 14 *It would be interesting to connect the result more closely to the results, in particular to Thm. 4.6 (ii), of [8], where all elements in $G(\mathbb{R} \times S)$ such that ∂_t is a Killing vector field, are classified in terms of completeness properties of associated Finsler type metrics on S, which are in particular satisfied for our example for the second item.*

Proof: Obviously $PT(M) \subset PC(M)$, so for the first assertion it is enough to see that $PC(M)$ is closed. Take any $g \in \mathrm{Lor}(M) \setminus PC(M)$. For every causal vector $v \in T_pM$ there exists a closed loop c_v at p such that the g-parallel transport along c_v does not fix v, that is $v \neq P^g_{c_v}(v)$. It is easy to see that still $v \neq P^h_{c_v}(v)$ for h in an open neighborhood of g, where now v may or may not be an h-causal vector. Associated to v we can take a tuple (W_v, V_v) consisting of open neighborhoods of g and v respectively, small enough such that $u \neq P^h_{c_v}(u)$ for every $h \in W_v$ and $u \in V_v$. The set L_g of g-causal vectors in T_pM itself is not compact, however, for every auxiliary scalar product in T_pM and associated norm $|\cdot|$, we can consider its unit sphere $S^k_pM := \{v \in T_pM \ / \ |v| = 1\}$, so $L_g \cap S^k_pM$ is compact, and therefore covered by a finite number of open sets $V_{v_1}, ..., V_{v_k}$. Take an open set $W \subset \cap_{i=1}^k W_{v_i}$ such that $g \in W$ and $V_{v_1}, ..., V_{v_k}$ still cover $L_h \cap S^k_pM$ for every $h \in W$. If $h \in W$ and $v \in L_h$, there exists i such that $\frac{v}{|v|} \in V_{v_i}$ so $P^h_{c_{v_i}}(v) \neq v$ because $h \in W \subset W_{v_i}$. This shows that $W \subset \mathrm{Lor}(M) \setminus PC(M)$.

Now for the second part, assume $(M, g) := (\mathbb{R}_t \times S, \alpha \otimes dt + dt \otimes \alpha + \overline{g})$ for a complete metric $\overline{g}$ on S and a $\overline{g}$-bounded nowhere vanishing one-form α on S (existence of such a form is well-known to be equivalent to vanishing of the Euler characteristic of S). Furthermore assume that there is a point $x \in S$ with sectional curvature $k^S_x > 0$. This can be done with an arbitrarily small perturbation of a given metric in the C^k-compact open topology.

Define

$$c(r) := -(1-r)dt^2 + r(dt \otimes \alpha + \alpha \otimes dt) + \overline{g}$$

for $r \in [0,1]$ which is a continuous curve in $Lor(M)$, smooth w.r.t. every C^k-compact-open topology. One finds that t is a Cauchy time function for all r. In fact, it is easy to see that any future vector v has positive scalar product with $\mathrm{grad}_{c(r)}(t)$.

Let $k : \mathbb{R} \to M$ be a causal curve. Now, if $t \circ k$ is bounded, it has a limit t_0 due to its monotonicity.

Now we parametrize k according to t, that is, $k(t) = (t, \overline{k})$, on a bounded interval $[0, b)$. The Cauchy-Schwarz inequality implies that $|\alpha(\overline{k}')| \leq |\alpha|\,|\overline{k}'|$, the norm always being the one defined by $\overline{g}$.

Then, for $r = 1$, using the causal character of k, we get $|\overline{k}'| \leq 2\,|\alpha|$.

In case of $r < 1$ we can solve the corresponding quadratic inequality for $|\overline{k}'|$ and get as a condition necessary for c causal

$$|\overline{k}'| \leq r\,|\alpha| + \sqrt{r^2\,|\alpha|^2 + (1-r)}.$$

Thus, by completeness of $\overline{g}$, also the S-coordinate along k has a limit at b, thus t is Cauchy, so $g_r \in G(M)$. Moreover, as $\mathrm{grad}_{c(r)}(t)$ is $c(r)$-parallel, in particular it is $c(r)$-Killing, we have $c(r)(\mathrm{grad}_{c(r)}(t), k')$ is constant along any geodesic k. Thus $c(r)$ is a causally geodesically complete metric. This (and the fact that $\mathrm{grad}_{c(r)}(t)$ is timelike for every $r \in [0,1)$ and lightlike for $r = 1$), shows that $c([0,1)) \subset F$ and $c(1) \in E$.

Suppose now that $c(1) \in PT(M)$, that is, there exists a timelike $c(1)$-parallel vector field $Z \in \mathfrak{X}(M)$, in particular it is linearly independent to $\frac{\partial}{\partial t}$ at any point. So there are $c(1)$-degenerate planes π in T_qM for any point $q \in M$ such that its null sectional curvature is zero, but this is not possible at points $p = (t,x) \in M$ for any $t \in \mathbb{R}$ because by hypothesis $\sec^S_x > 0$, see [15, Theorem 6.3 and Lemma 5.2]. Contradiction. ■

Now let us consider more closely the fine topologies. We want to argue in the following that they are *not* appropriate to consider spaces of metrics of precompact holonomy. For a finite-dimensional bundle $\pi : E \to M$, let $\Gamma^k(\pi)$ denote the space of sections of π of regularity C^k. The C^0-fine topology on $\Gamma^0(\pi)$ has as a neighborhood basis of a section f the family of sets $W_U := \{\gamma \in \Gamma^0(\pi)\ /\ \gamma(M) \subset U\}$ where U is an open neighborhood of[1] $f(M) \subset E$. If π is a vector bundle such that the fibers are locally convex metric vector spaces with an arbitrary translational-invariant metric[2] then we can describe the topology in a different manner: Let P be the space of smooth positive functions on M, then, for $p \in P$, which could be called a *profile function*, we set

$$U_p := \{f \in \Gamma^0(\pi)\ /\ d(f(x), 0_x) < p(x)\}$$

where 0_x is the zero in $\pi^{-1}(x)$. Then $\{f + U_p\}$ is a neighborhood basis for f as well. The equivalence of these two descriptions is easy to see, the arbitrariness of the auxiliary metric is compensated by the flexible choice of the profile function. The C^k-fine topology is defined by applying the same to the map $d^k\gamma$ as a section of the bundle $S^kE \to S^kM$ where, for a manifold N, S^kN is the bundle of unit vectors in T^kN for an arbitrary auxiliary Riemannian metrics. For more details cf. [4] and the references therein. The choice of the fiber metric used is irrelevant due to local compactness of the fiber.

[1]This notation might be slightly unfamiliar to the reader, recall, however, that a section f corresponds to a subset of E with certain properties, very much like a map between two sets A and B is a subset of $A \times B$.

[2]Keep in mind that here we use the word 'metric' not in the sense of bilinear form but in the sense of distance on a metric space.

The following theorem should be well-known to the experts, however we could not find any reference in the literature and thus include a proof here:

Theorem 15 *Let $\pi : E \to M$ be a metric vector bundle with locally convex fibers over a finite-dimensional manifold. Let a, b be two k-times continuously differentiable sections of π. Then a and b are in the same path connected component of $\Gamma^k(\pi)$ if and only if $\mathrm{supp}(a-b)$ is compact.*

Proof: As everything is translationally invariant, w.l.o.g. we can assume $b = 0$, the zero section. Assume the opposite of the statement of the theorem, that is, there is a noncompactly supported section a in the same path connected component as 0. By assumption, there is a C^0 curve $c : [0,1] \to \Gamma^k(\pi)$ from 0 to a. Choose $p_n \in \mathrm{supp}(a)$, $p_n \to \infty$ (a sequence leaving every compact set) and define $d_n := d(a(p_n), 0) > 0$. Let $(C_n)_{n \in \mathbb{N}}$ be a compact exhaustion with $p_n \in C_{n+1} \setminus C_n$. And consider an open neighborhood W_U of 0 as above with $U \cap \pi^{-1}(M \setminus C_n) \subset B_{d_n/n}$, for all n. As $c([0,1])$ is compact, it has a finite covering by sets of the form $U_i := c(t_i) + U$, say $U_1, ...U_m$. Then iterative application of the triangle inequality implies that $d(a(p_i), 0) \leq m \cdot d_i / i < d_i$ for $i > m$, contradiction. ■

Now, the first corollary of the previous theorem is that $G(M)$ alone has uncountably many path connected components each of which is intersected nontrivially by $PT(M)$. This holds even if we mod out the action of the diffeomorphism group on the space of metrics as it leaves the topology of the Cauchy hypersurfaces unchanged.

Corollary 16 *Within each path connected component of $G(M)$ in $\mathrm{Lor}(M)$ equipped with the C^0-fine topology, the topology of the Cauchy surface does not vary. Consequently, for M diffeomorphic to $\mathbb{R}^n$ with $n \geq 4$, the set $G(M)/\mathrm{Diff}(M)$ has uncountably many path connected components, each of which contains elements of $PT(M)$.*

Proof: For the first assertion, single out two metrics $g_1, g_2 \in G(M)$ in the same path connected component, then apply the previous theorem to $\mathrm{Lor}(M)$ (equipped with any auxiliary Riemannian metric on the fibers) to obtain that $g_1 = g_2$ outside of a compact set K. Now, any Cauchy surface which does not intersect K is a Cauchy surface for either metric. A recent result of Chernov-Nemirovski ([11], Remark 2.3) states that for an open contractible manifold C of dimension $n-1$, the product $\mathbb{R} \times C$ is diffeomorphic to $\mathbb{R}^n$. Now equip C with a complete metric g and consider the standard static manifold over (C, g). It is obviously diffeomorphic to $\mathbb{R}^n$. The Cauchy surfaces, however, are diffeomorphic to C. As we know (see [27] and the references therein) that for $n - 1 \geq 3$, there are uncountably many pairwise non-diffeomorphic contractible open manifolds (the Whitehead manifold being an example for $n-1 = 3$), the statement follows. ■

At this point, the reader probably is tempted to allow for an additional compact factor N to $\mathbb{R}^n$ and then to repeat the proof above. However, this is not possible as the proof would yield $C_1 \times N \cong C_2 \times N$ which could be true even for C_1 not homotopy equivalent to C_2, for an example with $N = \mathbb{S}^1$ see [9]. However, we think that it should be possible to use the argument above for any noncompact Cauchy surface, by replacing Whitehead's manifold suitably.

Remark 17 *It is well-known (see e.g. Corollary 7.32 and 7.37 in [4]) that causal completeness and causal incompleteness are C^1-fine-stable properties, i.e., given a globally hyperbolic causally complete resp. causally incomplete metric g, there is a C^1-fine open neighborhood U of g such that all metrics $h \in U$ are causally complete resp. causally incomplete. Using connectedness arguments we get easily that each connected component of a globally hyperbolic metric g_0 in the C^1-fine topology either consists entirely of causally complete or consists entirely of causally incomplete metrics.*

The following corollary states that the C^0-fine topology is already too fine for our purposes, as it isolates geometrically different metrics from each other. Namely, if we focus on one of the uncountably many path connected components, the result is only one $\mathrm{Diff}(M)$-orbit.

Corollary 18

1. *If $g_0 \in PT(M) \cap G(M)$ is timelike complete, then any timelike complete metric in the path connected component of g_0 in the C^0-fine topology is isometric to g_0.*
2. *If $g_0 \in PT(M) \cap G(M)$ is timelike complete, then any metric in the path connected component of g_0 in the C^1-fine topology is isometric to g_0.*

Proof: Let g_1 be another element of $PT(M) \cap G(M)$ path-connected to g_0. Both g_0 and g_1 admit global decompositions $I_0 : (M, g_0) \to \mathbb{R} \times (S, h_0)$ and $I_1 : (M, g_1) \to \mathbb{R} \times (S, h_1)$ (taking into account that the Cauchy surfaces of g_0 are diffeomorphic to those of g_1 following Corollary 16) with corresponding parallel vector fields $P^0 = \mathrm{grad}^{g_0}(t_0)$ resp. $P^1 = \mathrm{grad}^{g_1}(t_1)$ for associated temporal functions t_0 resp. t_1. As the metrics coincide outside of a compact set K following Theorem 15, the vector field P^0 is g_0-parallel *and* g_1-parallel on $(M \backslash K, g_1)$. Let us define $b := \sup\{t_0(x) \;/\; x \in K\}$, then P^0 is in particular g_1-parallel and g_1-timelike on $t_0^{-1}((b, \infty))$.

Choose $a > b$. We want to construct an isometry between (M, g_1) and $(\mathbb{R} \times S, -dt^2 + h)$ where h is the metric on $S := t_0^{-1}(a)$ induced by the metric g_1 (or equivalently, by the metric g_0, as the two are equal on $t_0^{-1}((b, \infty))$. Now we first show that $t_0^{-1}(a)$ is a Cauchy hypersurface of (M, g_1). So let a C^0-inextendible g_1-future curve c be given. By the usual non-trapping arguments, $c^{-1}(K)$ is compact. Let s denote its maximum, then $c|_{(s,\infty)}$ is a g_1-future causal curve in $M \setminus K$ that is also a g_0-future causal curve as $g_1|_{M \setminus K} = g_0|_{M \setminus K}$. As moreover $\lim_{r \to s} c(r) \leq b < a$, we conclude that the image of $c|_{(s,\infty)}$ intersects $t_0^{-1}(a)$. Thus, indeed, $t_0^{-1}(a) =: S$ is a Cauchy surface for (M, g_1).

We define a vector field Q^0 by parallel transport from S along P^1 with initial value P^0 on S, that is, $\nabla^1_{P^1} Q^0 = 0$ and $Q^0|_S = P^0$ (where $\nabla^1 := \nabla^{g_1}$). This indeed defines a vector field on M, as S is a g_1-Cauchy surface and P^1 is a complete future timelike vector field, thus its integral curves are C^0-inextendible future causal curves. Now we want to show that $\nabla^1 Q^0 = 0$. So let e_i be the P^1-parallel extension of a local orthonormal basis of TS, then we have $\nabla^1_{P^1} e_i = [e_i, P^1] = \nabla^1_{e_i} P^1 = 0$. Moreover, the mixed curvature terms vanish: $R^1(P^1, W)V = 0$ for any vectors V, W. Consequently, we get

$$\nabla^1_{P^1}\nabla^1_{e_i} Q^0 = R^1(P^1, e_i)Q^0 + \nabla^1_{e_i}\nabla^1_{P^1} Q^0 + \nabla^1_{[P^1, e_i]} Q^0 = 0,$$

and the initial condition $Q^0|_S = P^0$ implies $\nabla^1_{e_i} Q^0|_S = 0$, so the claim $\nabla^1 Q^0 = 0$ follows. Now, as Q^0 is g^1-parallel and g^1 is timelike complete, the flow of Q^0 is complete, any integral function of Q^0 and the flow of Q^0 define an isometry between (M, g_1) and $(\mathbb{R} \times S, -dt^2 + h)$ (note that S is always a level set for any integral function of Q^0 as Q^0 is orthogonal on S).

In fact, if $\Phi : \mathbb{R} \times S \to M$ is the flow of Q^0, identifiying $\{0\} \times S$ with $S \subset M$, using that Q^0 is g_1-timelike, complete and S a Cauchy hypersurface for g_1, it is clear that Φ is a diffeomorphism. Using that Q^0 is g_1-parallel it is clear that Φ is an isometry. Moreover, it sends $\frac{\partial}{\partial t}$ to Q^0.

As the same is true for g_0 via the function t_0, the metrics are isometric.

The second assertion follows from the first part and from the observation above that the whole path connected component of g_0 consists of causally complete metrics, see Remark 17. ■

It remains as an interesting question to examine the topology of subsets of globally hyperbolic metrics with holonomy of certain kinds for other manifold structures on $\mathrm{Lor}(M)$, possibly not coming from a vector space topology on $\mathrm{Bil}(M)$.

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