An Invitation to Lorentzian Geometry

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Abstract

The intention of this article is to give a flavour of some global problems in General Relativity. We cover a variety of topics, some of them related to the fundamental concept of *Cauchy hypersurfaces*: (1) structure of globally hyperbolic spacetimes, (2) the relativistic initial value problem, (3) constant mean curvature surfaces, (4) singularity theorems, (5) cosmic censorship and Penrose inequality, (6) spinors and holonomy.

Keywords: Global Lorentzian Geometry, Cauchy hypersurface, global hyperbolicity, Einstein equation, initial value problem, CMC hypersurface, singularity theorems, ADM mass, cosmic censorship hypotheses, Penrose inequality, spinors, Lorentzian holonomy.

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1 Introduction

The ground-breaking discovery of the theory of relativity has revealed that the physics of gravity can be described successfully by a theory treating space and time on the same footing, distinguished by the sign of an inner product. Nowadays, the mathematical framework of General Relativity can be regarded as a branch of Geometry (Lorentzian Geometry), in a similar sense like mathematics of Theoretical Mechanics are a branch of symplectic geometry. Admittedly, the physical leading ideas behind the geometric results are more subtle and less evident in General Relativity than in Mechanics. But after getting some familiarity with it, a new geometric world opens up, including unexpected new solutions (and problems) in *Riemannian* Geometry. We want to provide a brief overview of this wonderful world — which might be closer to the reader's field of research than (s)he may expect. We focus on global problems, which are usually the most interesting for mathematicians. We hope that this article will be also of some interest for physicists, which,

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frequently, are very familiar with local Differential Geometry, while global problems are neglected as of little importance for experimental purposes. Nevertheless, global questions can provide the necessary framework to the full theory and have implications in more practical issues — prominent examples supporting this claim are the definition of mass and energy or the Aharonov-Bohm effect.

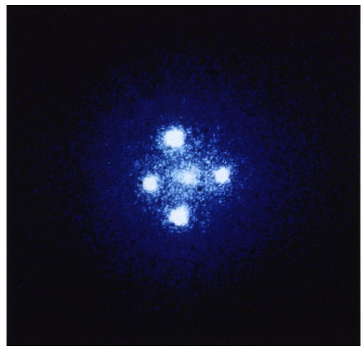

Figure 1: Einstein's cross, a gravitationally lensed region at 8 billion light years from earth, in the constellation Pegasus. A central galaxy focuses the light emanating from a quasar behind it producing a fourfold image of it. The images can be attributed to a single quasar via comparing characteristic spectra and synchronized relativistic jets. Gravitational lensing is a physical phenomenon closely related to the Lorentzian topic of Morse theory of causal curves [94] (see Section 6). Image taken by the European Space Agency's Faint Object Camera on board NASA's Hubble Space Telescope in 1990

One can distinguish different directions in Lorentzian Geometry:

1. *Looking at Riemannian Geometry.* That is, one tries to adapt the familiar Riemannian tools and results to the Lorentzian case, as far as possible. This was the case at the beginning of General Relativity, and is also the typical starting point for a standard mathematician — who has studied Riemannian Geometry but not Lorentzian geometry. This is not as straightforward as it sounds, because Lorentzian and Riemannian geometries, in spite of sharing common roots, separate fast in both aims and methods.

2. *Developing specific Lorentzian tools.* Concepts such as causality, boundaries, conformal extensions (Penrose diagrams), asymptotic behaviors (spatial and null infinities) or black holes, are specific to Lorentzian Geometry, without any analog in the Riemannian case. Here, physical intuitions are a very important guide, but we emphasize that these concepts have a completely tidy mathematical definition.

3. *Feed back to Riemannian Geometry.* Sometimes, a problem in Lorentzian Geometry admits a full reduction to a purely Riemannian problem. This

problem may be unexpected from a Riemannian approach, but now it becomes natural. The initial value constraint equations for Einstein's equation, the positive mass theorems (which yield the last step in the solution to the Yamabe problem!) or the Penrose inequalities provide remarkable examples of this situation. The reader will be able to appreciate that all the main results in Section 7 are stated in a purely Riemannian way, even though some motivations in this section, as well as further developments in Section 8, show the power and beauty of the bigger Lorentzian world.

In what follows, after a preliminary comparison between Lorentzian and Riemannian Geometries, a short overview of six research areas in Lorentzian Geometry, most of them motivated by General Relativity, is provided. In our choice of problems, global hyperbolicity and Cauchy hypersurfaces play an important role. The reason is twofold: on one hand, they play a central role in global problems, on the other hand, they are a very intuitive bridge between Riemannian and Lorentzian geometry (see Section 3). Additionally, the reader will find a variety of further topics, including hyperbolic equations, geodesics, CMC hypersurfaces, mass inequalities, holonomy and spinors. Some parts of this paper extend and update the earlier review [102].

2 From Riemannian to Lorentzian geometry

In this section, we briefly recall the basics of Lorentzian geometry and compare them to the Riemannian situation (see [17, 28, 48, 83, 89, 114] for further details). Let, throughout this article, M be an n-dimensional manifold, oriented if necessary. A *Lorentzian metric on* M is a symmetric bilinear form of signature $(1, n-1)$ at every point of M, that is, the maximal dimension of a negative definite subspace of T_pM is 1 and the maximal dimension of a positive definite subspace of T_pM is $n-1$. While on any manifold there is a Riemannian metric (using paracompactness), the same is not true any more if one replaces "Riemannian" by "Lorentzian": actually, the existence of a Lorentzian metric on a manifold M is easily seen to be equivalent to the existence of a one-dimensional subbundle of the tangent bundle which implies the vanishing of the Euler class of τ_M, or equivalently, of the Euler characteristic of M [15, Theorem 2.19]. For noncompact manifolds, however, this yields no obstruction: every non-compact manifold carries a Lorentzian metric.

In General Relativity, information is allowed to travel not along arbitrary curves, but only along *causal* ones. The notion of causality is first defined on the tangent spaces via the sign appearing in the Lorentzian metric. Namely (following the convention in [83]), a non-zero tangent vector $v \in T_pM$ is *causal* if it is either *timelike*, i.e. $g(v, v) < 0$, or *lightlike*, i.e. $g(v, v) = 0, v \neq 0$. A vector is called *spacelike* if $g(v, v) > 0$ (in particular, this convention means that $v = 0$ is non-spacelike and also non-causal). Both lightlike vectors and the 0 vector are called *null vectors*.

Let J_g be the set of all the causal vectors and I_g the set of timelike vectors. Then it is easy to see that each of $I_g \cap T_pM$ and $J_g \cap T_pM$ have two connected components. A connected Lorentzian manifold (M, g) is called *time-orientable* if J_g is disconnected; in this case, J_g has exactly two connected components (a fact that can be seen easily by general connectedness arguments using lifts of curves into open fibers). A *time-orientation* is a choice of one component J_g^+, which is called the *causal future* then (and its J_g-complement $J_g^- := -J_g^+$ is called *causal past*). Analogous considerations are valid for I_g. By a *spacetime*, we mean a (connected) time-oriented Lorentzian manifold (M, g) (the choice of time-orientation is not denoted explicitly).

The notion of "future" is then transferred from TM to M by curves: A (continuous) piecewise C^1 curve $c : I \subset \mathbb{R} \to M$ is called *future-directed causal* (resp. *future-directed timelike future*) iff $c'(t) \in J_g^+$ (resp. $c'(t) \in I_g^+$) for all t in each closed subinterval $I_j \subset I$ where c is C^1 — and correspondingly for the past. Then, we define the *chronological future* of p as:

$$I_g^+(p) := \{q \in M | \ \exists \ \text{ future-directed timelike curve } c \text{ from } p \text{ to } q \ \}.$$

Analogously, the *causal future* $J_g^+(p)$ is defined replacing "timelike" by "causal" in previous definition, and adding by convention $p \in J_g^+(p)$. There are natural dual 'past' notions, and the subscript g is removed when there is no possibility of confusion. Trivially, $I^\pm(p)$ are always open, but simple examples show that, in general, $J^\pm(p)$ are neither open nor closed — however, $\overline{I^\pm(p)} = \overline{J^\pm(p)}$ always holds. A state of a classical relativistic system at a point p can then only depend on events in the causal pastof p, and the state at p in turn can only influence the physics in the causal future $J^+(p)$ of p. The distinction of curves by their causal character corresponds to different elements of physical reality: massive bodies (e.g. observers) are supposed to travel along timelike curves, massless particles along lightlike curves, whereas spacelike curves do not admit a direct physical interpretation.

Many constructions in semi-Riemannian geometry are independent of the signature of the metric. First of all, any (non-degenerate) metric determines always a unique metric torsion-free connection (Levi-Civita connection). The Riemannian curvature tensor is defined in exactly the same way, and geodesics are defined as ∇-autoparallel curves; in particular, local convexity (in the sense of existence of a geodesically convex neighborhood around every point) is ensured. In Lorentzian signature, timelike geodesics are local maxima of an appropriately defined length functional (see below) on causal curves, and this property can be extended to lightlike geodesics, even though some relevant subtleties arise (see, for example, [82, Section 2]). In the framework of relativistic physics, the "observer" corresponding to a future-directed timelike curve c, is considered to be falling freely when c is a geodesic. Future-directed lightlike geodesics are regarded as "(trajectories of) light rays". No good interpretation holds for spacelike curves, even if they are geodesics, except for very special classes of spacetimes.

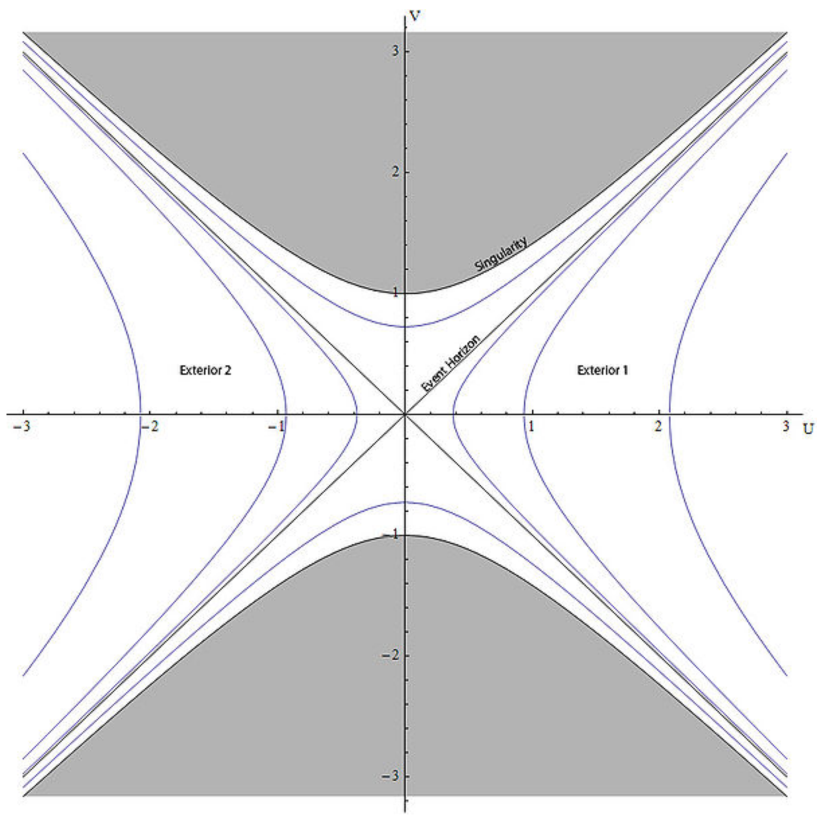

Figure 2: An instructive example: Schwarzschild-Kruskal metric. The two-dimensional Kruskal metric h is conformally equivalent to the standard Minkowski metric on the complement of the shaded region (in particular, at every point, the future causal cone $J_g^+ \cap T_pM$ is just the upper quadrant between the translates of the diagonals). The 4-dimensional physical spacetime is the warped product $h \times_{r^2} g_{\mathbb{S}^2}$. Lines through the origin are level sets of the coordinate t which is a time function for the exterior of the event horizon, whereas the displayed hyperbola correspond to constant r coordinate and are Killing orbits, timelike outside and spacelike inside the event horizon. The Schwarzschild part is exactly the future of the line $U = -V$; its exterior part describes successfully the gravitational field of an approximately spherically symmetric mass, e.g. of the sun. All timelike lines in the interior part have finite length (Image taken from http://commons.wikimedia.org/wiki/File:KruskalKoords.jpg licensed under the Creative Commons Attribution 3.0 Unported license by Author AllenMcC.).

Many of the Riemannian constructions carry over to the Lorentzian case *mutatis mutandis*. However, it is worthwhile to give a brief overview of some of the most important differences between Riemannian and Lorentzian geometry.

1. **For any given manifold M, the set of all the Lorentzian manifolds on M is not convex.** Recall that the set of Riemannian metrics on M is a (non-empty) convex cone. But this does not hold for Lorentzian metrics, even when there is no topological obstruction for their existence: one can check it trivially in dimension 2 (if g is Lorentzian then $-g$ is Lorentzian too), and that counterexample can easily be extended to higher dimensions. This fact makes several constructions of interpolation between different metrics significantly more complicated, e.g. in the theory of spinors.

2. **Sectional curvature is defined only for "non-lightlike" planes.** The reason is based on the fact that when the restriction of the metric g to some tangent plane $\pi \subset T_pM$ becomes degenerate, then the denominator in the definition of sectional curvature is 0. This fact also has the consequence that, if the sectional curvatures of the non-degenerate planes at p are not constant, then the sectional curvatures at p will reach values on all $\mathbb{R}$ (recall that one would divide by arbitrarily small positive and negative quantities, depending on the plane), see [89]. Also due to elementary algebraic reasons, a bound of the type $\mathrm{Ric}(v,v) > cg(v,v)$ for all $v \in T_pM \setminus 0$ and some $c \in \mathbb{R}$ cannot hold, and inequalities for curvatures must be understood in the sense of Andersson and Howard, [6]. In particular, the so-called *energy conditions* (see Section 4) will play the role of classical curvature bounds in some Lorentzian results.

3. **There is no natural embedding of the category of Lorentzian manifolds into metric measure spaces.** Recall that, in order to understand collapse processes and prove finiteness results like the ones due to Cheeger-Gromov, one needs to embed the manifolds and their possible limits into some space of metric spaces, independent of further specializations like Alexandrov spaces, CAT(0) spaces, etc. There is a well-known natural embedding of the category of Riemannian manifolds and convex open isometrical embeddings to the category of metric measure spaces which commutes with the forgetful functor to the category of topological spaces (just by taking the geodesic distance and the associated volume form). That is to say, from a Riemannian metric one can construct in a natural way a metric space compatible with the given topology. In contrast, there is no such construction in the Lorentzian case (which is easily seen by the existence of boosts around p in Lorentz-Minkowski space $\mathbb{L}^n := \mathbb{R}^{1,n-1}$ mapping an arbitrary point in $J^+(p) \setminus I^+(p)$ into an arbitrarily small neighborhood of p). Consequently, all the attempts of repeating the above compactness and finiteness results are doomed to failure in the Lorentzian regime. The same holds for most of the theory of isoperimetic inequalities. See, however, the the notion of "Lorentzian distance" below.

4. **The isotropy group of a point may be non-compact**. If a group G acts faithfully and isometrically on a Riemannian manifold (M, g), then the isotropy group of any $p \in M$, being a closed subgroup of $O(n)$, must be compact. This does not hold by any means in the Lorentzian case, as the Lorentz group $O_1(n) \equiv O(1, n-1)$ (which is the isotropy group of any $p \in \mathbb{L}^n$) is not compact. This represents an additional difficulty in finding invariant quantities (which are usually found as an average w.r.t. an associated Haar measure).

5. **No analog to Hopf-Rinow holds**. Geodesic dynamics change drastically: as there is no metric space associated to a given Lorentzian metric, none of the assertions of classical Hopf-Rinow theorem hold in Lorentzian geometry. Geodesic completeness neither implies b.a.-completeness (i.e., the completeness of curves with bounded acceleration) as in the Riemannian case. Neither compactness nor homogeneity of a Lorentzian manifold implies its geodesic completeness (however, by an argument due to Marsden [80], one knows that both properties together do). A counterexample for homogeneity is just a half-plane $H := \{(x_0, x_1) \in \mathbb{L}^2 | x_0 > x_1\}$ (the isometry group include the actions of translations by $\mathbb{R}(\partial_0 + \partial_1)$ plus the so-called boosts, i.e., the connected part of the identity of $O(1, 1)$). For compactness, consider the projection of the metric $g := -2dudv + (cos^4(v) - 1)du^2)$ on the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. Putting $u := x + t, v := x - t$ the g-geodesic $t \mapsto (1/t - t, \arctan(t))$ is then incomplete. Geodesics in Lorentzian manifolds can behave in a strange way for the Riemannian intuition. For example, there are inextensible lightlike geodesics $c : I \to M$ that are closed (in the sense that $c(I) = c(J)$ for a compact subinterval J of I) but they are non-periodic (they appear in the quotient of H above by a discrete subgroup of boosts generated e.g. by $(u, v) \mapsto (2u, v/2)$ where u, v are some natural lightlike coordinates). Moreover, conjugate points along a spacelike geodesic may accumulate [65, 95].

6. **Distinction between irreducibility and indecomposability for isometric actions**, as a Lorentzian vector space is not the sum of a (degenerate) subspace and its orthogonal. Recall that for an isometric Riemannian action ρ on T_pM, the orthogonal of an invariant subspace A of ρ, is invariant itself and complements A. The latter ceases to be true in the Lorentzian case if the metric restricted to A is degenerate. So, the action of a isometry group may be reducible to A, but the vector space T_pM may be not decomposable as sum of irreducible parts. This applies to holonomy representations, and made the discovery of the Lorentzian Berger list (the classification of all possible holonomy groups) considerably more difficult than the one of its Riemannian predecessor. This landmark was completed in 2005, cf. Section 8.

At this point, Riemannian geometers should not feel scared off by the above list of differences as they are balanced by a row of nice features listed below. First of all, the Laplacian can be formally defined in the Lorentzian setting as in the Riemannian one, but now it is an hyperbolic operator (d'Alembertian) and, even more, *Lorentzian Geometry is a natural framework to study hyperbolic equations, as Riemannian Geometry is a natural setting for elliptic ones.* This claim is supported by several geometric tools available in the Lorentzian setting but not in the Riemannian one. Let us point out some of these genuinely Lorentzian tools.

1. **Causality allows to visualize the conformal structure of spacetimes.** In fact, the datum $J_g^{\pm}$ which defines the causal structure is conformally invariant and, conversely, two Lorentzian metrics g, g' are (pointwise) conformal (i.e., $g' = \Omega g$, $\Omega^2 > 0$) if they have equal causal cones. One can define the *chronological* $\ll$ and *causal relations* $\leq$, namely $p \ll q$ iff $p \in I^-(q)$, $p \leq q$ iff $p \in J^-(q)$. Locally, either of these relations characterizes the conformal structure. So, *Causality* can be identified to conformal geometry in Lorentzian signature (nevertheless, a subtler modification of the notion of Causality has been recently introduced by García-Parrado and Senovilla [57], see also [56]). Remarkably, one can associate a (conformally invariant) *causal boundary* to every sufficiently well behaved spacetime, this allows to describe the possible asymptotics of timelike curves in a subtle way, ordering them by inclusion of their pasts [50].

2. **Higher compatibility with conformal structures of lightlike pregeodesics.** It is easy to see that there are positive functions of Schwartz class on Euclidean $\mathbb{R}^2$ which do not leave, when used as a conformal factor, any pregeodesic invariant; here, a pregeodesic means a geodesic up to a reparametrization. In contrast, on a Lorentzian manifold, conformal factors leave always some pregeodesics invariant, namely *all the lightlike pregeodesics*. Moreover, conformal changes even preserve conjugate points on them plus their multiplicities, [83].

3. **Completeness and singularities.** Apart from the differences noted above, there are also new notions of completeness in Lorentzian geometry. Traditionally, one distinguishes (logically independent) weaker notions than geodesic completeness: spatial completeness, lightlike completeness and timelike completeness, depending on the causal character of the geodesics in question. Such a subdivision of completeness does not exist in Riemannian geometry and, even more, two stronger notions appear in the Lorentzian setting: the above mentioned *b.a.-completeness* and Schmidt's *b-completeness* [104] (any of them coincide with usual geodesic completeness in Riemannian geometry). As a remarkable difference with the Riemannian case, both, completeness and incompleteness are C^r unstable, for every $r \in \mathbb{N}$, even in the case of Lorentzian metrics on a torus [99], but some results on stability can be still obtained in the Lorentzian case, as C^1-fine stability in the globally hyperbolic case, see [17, 28]. The interplay between these particularities, causality and some physical interpretations, have the effect that, in Lorentzian Geometry, *singularity theorems* (which ensure incompleteness rather than completeness) play

an important role, see below.

4. **Reverse triangle inequality and Lorentzian "distance"**. As a construction related to Causality but not conformally invariant, one can define the *time separation* $d(p,q)$ between two points $p,q \in M$ of a spacetime (M,g) as the *supremum* of the lengths of the future-directed causal curves starting at p and ending at q; this supremum may be infinity, and it is regarded as equal to 0 if $p \not\leq q$ (in fact $d(p,q) = 0$ iff $p \not\ll q$). The corresponding function $d : M \times M \in [0,\infty]$ is commonly called the *Lorentzian distance*, as it satisfies a *reverse* triangle inequality (due to the existence of a reverse triangle inequality for causal vectors in the same cone in any T_pM) with some similarity to the Riemannian case. However, it presents also some big differences with the Riemannian case; for example, $d(p,q) > 0$ implies either $d(q,p) = 0$ or there exists a closed timelike curve through p and q (and, so, $d(p,p) = d(q,q) = \infty$). As a clear connection with Causality, $d(p,q) > 0$ iff $p \ll q$ and, under a mild Causality condition on the spacetime (strong causality, i.e., absence of "almost closed" causal curves) the Lorentzian metric g can be reconstructed from d, as in the Riemannian case. However, the interplay of d with Causality makes it a genuinely Lorentzian element. For example, a spacetime is globally hyperbolic (see definition below) iff the Lorentzian distance is finite and continuous for all the metrics in the conformal class of g (see for example [17, 83]).

3 Causality and global hyperbolicity

Good conditions on the causality of a spacetime may yield some connections between Riemannian and Lorentzian manifolds and links between hyperbolic and elliptic equations. A key notion is *global hyperbolicity* which is to be developed here and which will play a role in the spirit of *completeness* for Riemannian manifolds.

A spacetime (M,g) is called *globally hyperbolic* iff it is causal[1] (i.e. $p \notin J^+(p)$ for all $p \in M$) and diamond-compact. Here, "diamond-compact" means $J^+(p) \cap J^-(q)$ compact for all $p,q \in M$. The condition of causality corresponds to the existence of global solutions of natural linear differential operators for initial values on maximal achronal hypersurfaces while the condition of diamond-compactness corresponds to their uniqueness: consider the examples of a flat Lorentzian torus for non-existence and of a flat vertical strip in Minkowski space for non-uniqueness. In terms of physics, diamond-compactness corresponds to predictability of nature or Laplace's demon principle (which has been of some influence at least in classical physics), whereas causality corresponds to the exclusion of the possibility of time

[1]Typically, an a priori stronger hypothesis that causality is used to define global hyperbolicity, namely, *strong* causality. In [24] it has been shown that both notions agree. For simplicity, we renounce giving details here and simply use the more recent definition of global hyperbolicity instead.

machines[2]. The link between global hyperbolicity and Riemannian completeness comes from the following result, which lies in the spirit of Hopf-Rinow's (see for example [17, 89]):

Proposition 3.1 *In a globally hyperbolic spacetime* (M, g)*, the Lorentzian distance* d *is finite, continuous and satisfies the* Avez-Seifert property*, i.e., for any pair of causally related distinct points* $p, q \in M$*,* $(p \leq q)$ *there exists a causal geodesic from* p *to* q *with length equal to* $d(p, q)$*.*

There are many examples of globally hyperbolic manifolds:

1. A Lorentzian product $(\mathbb{I} \times N, -dt^2 + g_N)$ for an interval $\mathbb{I}$, is globally hyperbolic iff g_N is a a complete Riemannian metric on N; in particular, Lorentz-Minkowski spaces are globally hyperbolic.

2. *Narrower Cones Principle*: if (M, g) is globally hyperbolic and h is another Lorentzian metric on M with $J_h \subset J_g$, then (M, h) is globally hyperbolic as well; in particular, global hyperbolicity is conformally invariant. This implies that, instead of the Lorentzian products in item 1, one can equally consider *Generalized Robertson-Walker spacetimes* (with complete g_N) i.e, warped products $(\mathbb{I} \times N, -dt^2 + f(t)g_N)$ for some positive function f.

3. If (M, g) is globally hyperbolic and $A \subset M$ is a causally convex open subset of M in the sense that causal curves cannot leave and then re-enter A, then $(A, g|_A)$ is globally hyperbolic as well.

4. Using convex neighborhoods, it is easy to see that any point in any Lorentzian manifold has a globally hyperbolic neighborhood.

5. Global hyperbolicity of Lorentzian metrics is a C^0-fine stable property in the space of Lorentzian metrics which has been shown by the works of Geroch [64] (taking into account the progress made by Bernal and Sánchez in [22], see [103]); see also Lerner [75] or the extended version on arxiv.org of Benavides and Minguzzi [18].

This last point goes in the direction of Proposition 3.1, i.e. the role of global hyperbolicity is related to Riemannian completeness, as both properties are C^0 stable (but geodesic completeness is not C^r-stable for any r in the general Lorentzian case and only C^1-fine-stable for globally hyperbolic manifolds, as mentioned above).

[2] Time machines are in contradiction to the unspoken fundamental assumption of the free will of the experimentalist taken by the vast majority of physicists, in the sense that any observer, in contrast to physical nature around him, is assumed to be able to take decisions like preparing a spin-up or a spin-down state in a manner which is in principle unpredictable for others, compare the discussion of Bell's inequality and the EPR paradox. Note that without that assumption, time machines do not contradict any other principles of physics — with the possible exception of predictability of nature, cf. the article of Krasnikov [70] as well as its critical reception in [77].

Geroch [64] showed in 1970 that a spacetime is globally hyperbolic[3] if and only if it contains a Cauchy hypersurface, that is, a set Σ that is crossed exactly once by any inextensible timelike curve (a posteriori, Σ must be then a topological hypersurface, see [89]). Moreover, he gave a construction of a continuous *Cauchy time function* t, which means that t increases strictly monotonously along every causal future-directed curve, and is surjective onto $\mathbb{R}$ along any inextensible causal future-directed curve. His construction involved volumes of sets type $J^{\pm}(p)$ for a finite volume form. For a long time, it was not known, but generally assumed, that Σ could be taken as a smooth and spacelike (non-degenerate) hypersurface and, even more, that for such a prescribed Cauchy hypersurface Σ one could find even a *Cauchy temporal function* vanishing on Σ. This term is more special than the one before and denotes a *smooth* function t whose gradient satisfies $g(\text{grad}t, \text{grad}t) < 0$ with grad$t$ past-directed, plus the surjectivity property above. (Note that not all smooth Cauchy time functions are temporal: consider, on $\mathbb{L}^2$, the function $t(x_0, x_1) := (x_0 + x_1)^3$). Functions of this kind automatically lead to metric splitings, that is, they imply that the manifold is isometric to

$$(\mathbb{R} \times N, -f^2 \cdot dt^2 + g_t) \tag{1}$$

where $f > 0$ is a smooth function on $\mathbb{R} \times N$ and g_t is a smooth family of Riemannian metrics on the level sets of t, and all level sets of t are Cauchy. The interest in these questions is obvious: on one hand, (smooth) spacelike Cauchy hypersurfaces are the natural ones for initial data (Einstein equation, Penrose inequality..., see the next sections); on the other, the orthogonal splitting is useful for many properties: Morse Theory, quantization, to find global coordinates, etc. Moreover, it also leads to remarkable analytic results: not only adapted linear symmetric hyperbolic systems (that is, those given by a first-order differential operator on a vector bundle whose symbol is exactly positive-definite on I_g), enjoy global existence and uniqueness for arbitrary initial values at a Cauchy hypersurface, but the same is true for appropriate nonlinear equations like Yang-Mills equations, as shown by Chrusciel and Shatah [40]. Physically, each temporal function t determines in a natural way not only a one-parameter family of diffeomorphic "physical spaces" (the slices $t =$ constant), but also a Wick rotation, obtained by inverting the sign on $\mathbb{R} \cdot \text{grad}t$ and leaving the orthogonal complement unchanged.

Sachs and Wu [100, p. 1155] posed the existence of a *smooth* Cauchy hypersurface in any globally hyperbolic spacetime as a first open "folk" problem. Such a type of problems cannot be overlooked by physicists as minor questions of mathematical rigor, as the requirements in the definition of global hyperbolicity are plausible from the physical viewpoint, but the assumption of a splitting *a priori* of the spacetime as in (1) (the type of expression truly useful for several physical purposes) would be totally unjustified. In a series of papers published along 2003-2006, Bernal and

[3] He considered a different, but equivalent, notion of global hyperbolicity, based on the compactness of the space of causal curves connecting each two points, but this is not specially relevant at this point.

Sánchez [21, 22, 23] (see also [101]) gave a full solution by showing that a splitting as (1) can be obtained and, then, any prescribed spacelike Cauchy hypersurface can be chosen as the level $t = 0$ of the splitting; their proof used local convex coordinates patched together in a sophisticated way. There has been, however, quite a few of interesting developments since then. In 2011, Müller and Sánchez [88] solved the question of which Lorentzian manifolds are isometrically embeddable in some $\mathbb{L}^n$ (in the spirit of Nash's theorem). With this aim, they proved, in particular, that any globally hyperbolic spacetime admits a splitting as in (1) with an upper bounded function $f < 1$ (this yielded directly the isometric embeddability of all globally hyperbolic spacetimes). Further properties on both, the splitting (bounds for curvature elements of the slices, flexibility) and the isometric embedding in $\mathbb{L}^N$ (closedness) were obtained then by Müller [86, 87]. In 2012, Fathi and Siconolfi [49] proved the existence of a Cauchy temporal function in a class of geometric spaces with a *cone structure* (which generalized notably the class of globally hyperbolic spacetimes); their proof involves tools from weak KAM theory. By taking into account the progress along these decades (including old work by Seifert), Chrusciel, Grant and Minguzzi [38] have proved very recently that, for some appropriate non-canonical choice of a volume form, also the original functions defined by Geroch become C^1 (and can be smoothed out further by local convolutions). The interplay among these tools is an exciting matter of study [103].

Summing up from a broad perspective, classical elementary results as Proposition 3.1, deeper structural results as splitting (1), and links with other parts of Differential Geometry or Mathematical Physics (Morse theory, Geometric Analysis, Cosmic Censorship, Wick rotation, Einstein and Yang Mills equations...), show that Riemannian geometry is an indispensable tool in the theory of globally hyperbolic manifolds, but the study of the interplay between the two regimes has just been initiated.

4 Initial value problem

Einstein's field equation can be written (in suitable units) as

$$\mathrm{Ric} - \frac{1}{2} S g = 8\pi T. \tag{2}$$

Here, the geometric terms on the left hand side (Ricci tensor Ric, scalar curvature S) are related to a symmetric 2-tensor on the right hand side, the "stress-energy" T, which describes the distribution of matter/energy.

More properly, we must emphasize that the unknown quantity is not only the metric g (with Ric and S): equations for T must be added to get a coupled system with (2). Nevertheless, we will assume for simplicity (in addition to $\dim(M)$=4, when necessary) the following cases:

- Along this section, $T = 0$ (vacuum), i.e. (2) becomes Ric$\equiv 0$.

- In the next sections, solutions with T non-determined but satisfying only any of the (mild) "energy conditions" as: (1) Weak: $T(v,v) \geq 0$ for any timelike v (density energy is nonnegative), (2) Dominant: $-T(v,\cdot)^\flat \equiv -g^{ij}T_{jk}v^k$ is either future-directed causal or 0 for any future timelike v (energy flow is causal), (3) Strong: equivalent via Einstein equation to the timelike convergence condition, $\mathrm{Ric}(v,v) \geq 0$ for timelike v (gravity, on average, attracts).

The well-posedness of Einstein equation requires an input of initial data on a 3–manifold Σ which permits to obtain a ("unique, maximal") spacetime (and eventually a T) such that Σ is embedded in M consistently with the initial data. The problem is complicated: a classical theorem such as Cauchy-Kovalevskaya's is not applicable, and, even more, in principle the system of equations is not hyperbolic. Nevertheless, there exist a highly non-trivial procedure — based on the existence of *harmonic coordinates* — which allows one to find an equivalent (quasi-linear, diagonal, second order) hyperbolic system. The standard global result was obtained by Choquet-Bruhat and Geroch [35]:

Theorem 4.1 *Let (Σ, h) be a (connected) Riemannian 3-manifold, and σ a symmetric two covariant tensor on Σ which satisfies the compatibility conditions of a second fundamental form (Gauss and Codazzi eqns.) Then there exist a unique spacetime (M, g) satisfying the following conditions:*

(i) $\Sigma \hookrightarrow M$, consistently with h, σ (i.e., $h = g|_\Sigma$ etc.)

(ii) Vacuum: Ric$\equiv 0$ (this can be extended to any family T of natural divergence-free symmetric 2-tensors, e.g. to those coming from natural symmetric hyperbolic field theories).

(iii) Σ is a Cauchy hypersurface of (M, g).

(iv) Maximality: if (M', g') satisfies (i)—(iii) then it is isometric to an open subset of (M, g).

As suggested previously, the property (iii) becomes essential for the well-posedness of the problem — namely, the existence of a solution spacetime can be proven because no timelike curve crosses Σ twice, and the uniqueness because all timelike curves cross Σ at least once.

Remark 4.2 *(SCCC).* Even though the solution (M, g) provided by Theorem 4.1 is maximal, it may be extensible as a spacetime, that is, (M, g) may be isometric to an open proper subset of another spacetime $(\bar{M}, \bar{g})$ — even a vacuum one. In this case, Σ cannot be a Cauchy hypersurface of the extension, and two possibilities arise: (a) $(\bar{M}, \bar{g})$ is not globally hyperbolic or (b) the initial Σ was not "chosen adequately", as an input hypersurface for a whole physically meaningful spacetime. Thus, an important question is how to characterize the (in)extendibility of (M, g).

This question becomes extremely important in General Relativity because physical intuition suggests that spacetime is inextensible, but it suggests at the same time that it should be predictable from initial data and, thus, globally hyperbolic.

The *Strong Cosmic Censorship Conjecture (SCCC)* asserts that, for generic physically reasonable data (including a "good choice" of Σ), (M, g) is inextendible.

Of course, a non-trivial problem of the conjecture, is to explain carefully what "generic physically reasonable data" means.

A systematically studied problem is to characterize/classify the solutions of (vacuum) Einstein equation. By using Theorem 4.1, this is rather a purely Riemannian problem (roughly: given data as, say, (Σ, h), classify the σ's which satisfy Gauss and Codazzi equations). There are two specially important methods of solution (see [11] for a detailed exposition and [37] for updated references):

- *Conformal.* Initial data are divided into two subsets: a subset of *freely specified* conformal data (the conformal class of h, a scalar field τ, and a symmetric divergence free 2-tensor $\tilde{\sigma}$), and a subset of *determined* data (a function $\phi > 0$, a vector field $W \in \chi(\Sigma)$), which are derived from the free data by means of differential equations. The interpretation and equations for these data vary with two types of conformal method (the method (A) or semi-decoupling, whose origin goes back to Lichnerowicz [76], and the method (B) or conformally covariant). The problem is then to show if there exists solutions for the equations of the determined data, and classify them.
- *Gluing solutions.* As a difference with the conformal method, this is not a general one, but it is very fruitful in relevant particular cases. Corvino and Schoen [41, 42] glue any bounded region of an asymptotically flat spacetime with the exterior region of a slice of Kerr's — this case becomes specially interesting as the "no hair theorems" highlight Kerr spacetime at the final state of the evolution of a black hole. The useful gluing by Isenberg et al. ([68], see also the initial data engineering [39]) constructs consistent initial data for Einstein equation from the connected sum of previously obtained data (for example, construction of wormholes).

For the general conformal method, the results depend on different criteria — topology of Σ, asymptotic behaviour, regularity (analytic, smooth, Hölder class...), metric conformal class (Yamabe type)... The most important one is the mean curvature H. Essentially, when H is constant almost all is known (at least if $T = 0$); in fact, if Σ is either compact without boundary, or asymptotically flat or hyperbolic, it is completely determined which solutions exist (and they exist for all but certain special cases). When H is nearly constant there are many results, but also many open questions; otherwise, there are very few results.

5 Constant mean curvature spacelike hypersurfaces

The importance of (spacelike) hypersurfaces of constant mean curvature (CMC) H in a spacetime (M, g), has been stressed above in relation to the initial value problem, but they are also important for other issues in General Relativity (see the survey by Marsden and Tipler [79]). Here we will explain some results on their existence and uniqueness. We will restrict to the case when the hypersurface Σ is spacelike, as it is easy to see that a submanifold can extremize area only when its dimension is equal either to the index or to the coindex of the metric (otherwise, the area may be critical, but not extremal). Moreover, when $H = 0$ the spacelike hypersurfaces are either maximal or neither maximize or minimize area. Nevertheless, in both cases they are usually called *maximal*, just like Riemannian minimal surfaces. About the results on existence, we point out (see Gerhardt's book [62] for a detailed study):

- After the special case of Lorentz Minkowski (see below), the first natural problem to be considered is the construction of one CMC hypersurface or, in general, a spacelike hypersurface Σ with a *prescribed* mean curvature H, in a given spatially compact globally hyperbolic spacetime. A relevant result of global existence was due to Claus Gerhardt [59] in 1983, under the condition of existence of barriers. An upper resp. lower r-barrier is a closed spacelike achronal hypersurface with mean curvature $> r$ resp. $< r$. If there is an upper r-barrier Σ^+ and a lower r-barrier Σ^-, Gerhardt shows that there is a CMC hypersurface of mean curvature r in $I^+(\Sigma^-) \cap I^-(\Sigma^+)$. This is shown by solving the Dirichlet problem for a given boundary curve by making some a priori gradient estimates and, then, by applying a Leray-Schauder fixed point theorem. To enhance the constructiveness in the last part, Ecker and Huisken [45] used an evolutionary equation in terms of the mean curvature flow starting at some Cauchy hypersurface $\Sigma \subset I^+(\Sigma^-) \cap I^-(\Sigma^+)$ which provided a better control over the hypersurfaces — for example, it allows to fix all points of vanishing mean curvature during the process. Even though they had to assume some additional conditions (the timelike convergence condition and a more technical structural monotonicity condition), Gerhardt [60] refined Ecker and Huisken's flow method, showing that such additional conditions were unnecessary. The improved control allows to solve also related problems such as: (a) given a prescribed point in M, construct a CMC hypersurface passing through the point, or (b) given a compact Cauchy surface in M find a compact CMC Cauchy surface with the same volume.
- Removing spatial compactness, the next step is to consider the existence of CMC hypersurfaces in asymptotically flat spacetimes. Substantial contributions, specially in the maximal (eventually up to a compact subset) case, have been made by Bartnik [9], Bartnik, Chrusciel and Murchadha [10], and Ecker [44]. All three articles assume an energy inequality and, moreover, a connection between radial and time variables (called "uniform interior condition" in the first and "bounded interior geometry" in the second article). Roughly, both versions of the latter condition assert that the deviations from Minkowski geometry propagate with subluminal velocity — so one can expect the condition to be true for massive Klein-Gordon Theory, e.g. Again, the first article uses Leray-Schauder's fix point theorem, while the other articles use long-time convergence of the mean curvature flow, which provides a better control on the surfaces.
- The question of constructing a whole foliation by CMC hypersurfaces was

also studied. In the cited 1983 article by Gerhardt [59], he considered a globally hyperbolic spacetime with compact Cauchy hypersurfaces satisfying the timelike convergence condition. Under these hypotheses, slices with a CMC $H \neq 0$ are unique and, if there are two different maximal slices, then both have to be totally geodesic and the region enclosed by them must be static — thus, the existence of two different hypersurfaces of CMC implies strong obstructions. In fact, the mean curvature must increase monotonously in foliations by CMC hypersurfaces. The timelike convergence condition is replaced by the mere assumption of a lower bound to the Ricci tensor on timelike vectors in a second article by Gerhardt [61]. This article also treats exclusively the case of globally hyperbolic spatially compact spacetimes. Here, the statement is the following: If for some sequence of Cauchy surfaces Σ_n of M with $\Sigma_{n+1} \subset I^+(\Sigma_n)$ for all n and $\bigcap I^-(\Sigma_n) = M$ there is a sequence of n-barriers $B_n \subset I^+(\Sigma_n)$ for any $n \in \mathbb{N}$, then there is a Cauchy surface Σ of M such that $F := I^+(\Sigma)$ can be foliated by a CMC foliation and the mean curvature is a temporal function on F.

In the last item, some results on uniqueness of CMC hypersurfaces appear implicitly, but this question deserves a bigger attention. A neat problem on uniqueness can be stated as follows, see [1]. Consider a Riemannian n-manifold (M, g_R), a smooth positive function defined on some interval $f : I \subset \mathbb{R} \to \mathbb{R}$, and the solutions u to the differential equation on M:

$$\begin{gathered} u(M) \subset I, \qquad |\nabla u| < f(u) \\ \operatorname{div}\left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - |\nabla u|^2}}\right) = nH - \frac{f'(u)}{\sqrt{f(u)^2 - |\nabla u|^2}}\left(n + \frac{|\nabla u|^2}{f(u)^2}\right) \end{gathered} \tag{3}$$

for some constant H. The graphs of its solutions can be regarded as the spacelike[4] hypersurfaces of CMC equal to H in a Generalized Robertson-Walker spacetime $I \times_f M$ (recall, abusing of the notation $g \equiv -dt^2 + f(t)^2 g_R$). Notice that this *Calabi-Bernstein equation* is the Euler Lagrange one for the functional

$$\mathcal{A}(u) = \int_M f(u)^{n-1}\sqrt{f(u)^2 - |\nabla u|^2}\, dV$$

under the constraint $\int_M \left(\int_{u_0}^u f(t)^n\right) dV =$ constant. A specially relevant case of this equation was solved by Cheng and Yau [34] (after the solution by Calabi for $n \leq 4$):

Theorem 5.1 *The only entire solutions to Calabi-Bernstein equation* (3) *in* $\mathbb{L}^{n+1}$ *(i.e.,* $(M, g_R) \equiv \mathbb{R}^n$*,* $I \equiv \mathbb{R}$*,* $f \equiv 1$*) are linear (or affine) functions.*

As a consequence, the only complete maximal hypersurfaces in Lorentz-Minkowski space are the spacelike hyperplanes.

[4] Because of the gradient condition $|\nabla u| < f(u)$.

In fact, they proved that any maximal spacelike hypersurface which is also a closed subset in $\mathbb{L}^{n+1}$ is a hyperplane. This yields a surprisingly simple solution to the Calabi-Bernstein problem (recall, for example, that the analogous results change dramatically for minimal hypersurfaces with dimension larger than seven).

There were, however, well-known counterexamples for the case of CMC hypersurfaces with $H \neq 0$, [110, 112]. So, a line of results about the uniqueness of CMC hypersurfaces has appeared. By using integral inequalities, one can check that all compact CMC hypersurfaces Σ in any GRW spacetime under the *null convergence condition* (i.e., $\mathrm{Ric}(v, v) = 0$ on null vectors) are totally umbilical — and thus, under mild conditions, Σ is a slice $t =$constant. Such a type of result can be also generalized further, see [84, 1] and references therein. The feed-back of these results with Riemannian ones have been especially fruitful. Remarkably, both the GRW structure and the restriction for the hypersurfaces of being *spacelike*, yields simplifications that inspired some hypotheses for the Riemannian case.

Under some conditions, previous results can be extended to the case when Σ is complete but non-compact, [4]. In general, for the non-compact case, the so-called *Omori-Yau maximum principle* (or *asymptotic Cheng-Yau principle*), becomes useful. This principle is stated for complete, connected, noncompact Riemannian manifolds and, roughly speaking, means that any smooth function u bounded from above on M, will admit a sequence $\{x_k\} \subset M$ which plays the role of a maximum (say, $\lim_{k\to\infty} u(x_k) = \sup_M u$, $|\nabla u(x_k)| \leq 1/k$ and $\Delta u(x_k) \leq 1/k$). The principle holds when the Ricci curvature is bounded from below as well as in other more refined cases. We refer to [96, 3] and references therein for the recent progress on the Omori-Yau principle and its applications to hypersurfaces in both, Riemannian and Lorentzian Geometry.

It is also worth pointing out that, when dim(M)= 3 (i.e., the hypersurface Σ is a surface), new tools appear. For example, a different approach to the non-compact case has been developed very recently for CMC spacelike surfaces in certain 3-dimensional GRW spacetimes $I \times_f F$; the main idea is to prove that, under some natural assumptions, a metric conformal to the induced one on the surface Σ must be parabolic, see [98] and references therein. Recall also that the analog to the classical Björling problem (construct a minimal surface in $\mathbb{R}^3$ containing a prescribed analytic strip, solved by H. A. Schwarz in 1890) has been also considered in the Lorentzian case; this yields a representation formula for maximal surfaces and allows to construct new ones explicitly; see [2, 33] for the case of $\mathbb{L}^3$ and [85] for the general problem in arbitrary spacetimes, without restriction of the dimension.

6 Geodesics and singularity theorems

In some concrete spacetimes, singularities might be defined "by hand" but a general definition is difficult [63], for example:

1. The singularity will not be a point of the spacetime, but placed "at infinity" -but no natural notion of infinity exists in general.

2. The curvature tensor R is expected to diverge, but all its scalar invariants $(\sum R_{ijkl}R^{ijkl}, \sum \nabla_s R_{ijkl} \nabla^s R^{ijkl}, S...)$ may vanish when $R \neq 0$.

At any case, some sort of "strange disappearance" happens if the spacetime is *inextensible, but an incomplete causal geodesic exists*, and these two conditions will be regarded as *sufficient* for the existence of a singularity. Then, the aim of the so-called *singularity theorems* is to prove that causal incompleteness occurs under general natural conditions on T (an energy condition) and on the causality of the manifold, as global hyperbolicity. Nevertheless, recall that, rather than "singularity" results, they may be "incompleteness" ones: the physical conclusion of these theorems could be that a physically realistic spacetime cannot be globally hyperbolic, rather than being singular. So, they become "true singularity" results when an assumption as global hyperbolicity is removed... or if SCCC (Remark 4.2) is true!

Recall the following Hawking's singularity theorem (see [48] or [89] for a detailed exposition):

Theorem 6.1 *Let (M, g) be a spacetime such that:*

1. It is globally hyperbolic.

2. Some spacelike Cauchy hypersurface Σ strictly expanding $H \geq C > 0$ (H is the future mean curvature and expansion means "on average")

3. Strong energy (i.e., timelike convergence condition) holds: Ric$(v, v) \geq 0$ for timelike v.

Then, any past-directed timelike geodesic γ is incomplete.

Sketch of proof. The last two hypotheses imply that any past-directed geodesic ρ normal to Σ contains a focal point if it has length $L' \geq \frac{1}{C}$. Thus, once Σ is crossed, no γ can have a point p at length $L > \frac{1}{C}$ (otherwise, a length-maximizing timelike geodesic from p to Σ with length $L' \geq L$ would exist by global hyperbolicity, a contradiction). □

This result is very appealing from a physical viewpoint, because the assumption on expansion seems completely justified by astronomical observations. Remarkably, the hypothesis $H \geq C > 0$ for some constant C cannot be weakened into $H > 0$, as shown by a surprising (as physically realistic and far-from-vacuum) example due to Senovilla [108]. From a mathematical viewpoint, the reader can appreciate the isomorphic role of the hypotheses above with the typical ones in Myers type results, say: global hyperbolicity/ (Riemannian) completenes and timelike convergence condition/ positive lower bound on the Ricci tensor.

Singularity theorems combine previous ideas with (highly non-trivial) elements of Causality. Essentially, there are two types:

1. Proving the existence of an incomplete timelike geodesics in a global, cosmological setting.

   This is the case of Theorem 6.1, and some hypotheses there (specially glob. hyp.) are weakened or replaced by others. For example, Hawking himself proved that, if Σ is compact, global hyperbolicity can be replaced by assuming that Σ is achronal (i.e., non-crossed twice by a timelike curve). In this case, the timelike incompleteness conclusion holds, but in a less strong sense: at least one timelike incomplete geodesic exist.

2. Proving the existence of an incomplete lightlike geodesic in the (semilocal) context of gravitational collapse and black holes.

For the latter, the notion of (closed, future) *trapped* surface K (or $n-2$ submanifold) becomes fundamental. Its mathematically simplest definition says that K is a compact embedded spacelike surface without boundary, such that its mean curvature vector field $\vec{H}$ is future-directed and timelike on all K [107] –essentially, this means that the the area of any portion of K is initially decreasing along *any* future evolution; when it is only non-increasing, K will be said *weakly trapped*. Trapped surfaces are implied by spherical gravitational collapse. One would expect that, at least in asymptotically flat spacetimes (see next section), they must appear if enough matter is condensed in a small region and, under suitable conditions, must imply the existence of a *black hole* (see [43] and references therein). That is, the physical claim is that "gravitational collapse implies incompleteness", and a support for this claim is provided by the following Penrose's theorem (the first modern singularity theorem [91] -after the works by Raychaudhuri and Komar):

Theorem 6.2 *Let (M, g) be a spacetime such that:*

1. *Admits a non-compact Cauchy hypersurface.*
2. *Contains a trapped surface.*
3. *$Ric(k, k) \geq 0$ for lightlike k.*

Then there exist an incomplete future-directed lightlike geodesic.

As emphasized by Senovilla [109], the pattern of a singularity theorem has three ingredients: firstly, a bound on the Ricci curvature, secondly, a causality condition, and thirdly, an initial condition on a nonzero-codimensional subset. Remarkably, a unified treatment of both types of singularity theorems has been carried out recently by Galloway and Senovilla [55]. Singularity theorems are very accurate, even though it would be desirable to obtain general results on the nature of the incompleteness, or ensuring divergences of R in some natural sense. So, the finding of further types of singularity theorems would be very desirable for physical purposes [108, 109].

Remark 6.3 The subtleties of Lorentzian completeness also appear in the Lorentzian analogue of Cheeger-Gromoll theorem (see for example [17, Chapter 14]). To obtain the Lorentzian splitting, of a spacetime (M, g), $\dim(M) > 2$, as a product $(\mathbb{R} \times \Sigma, -dt^2 + g_\Sigma)$, where (Σ, g_Σ) is a complete Riemannian manifold, one imposes: (a) *either geodesic completeness or global hyperbolicity*, (b) the *timelike convergence condition* (as the meaningful weakening of the positive semi-definite character of the Ricci tensor in the Riemannian case), and (c) the existence of a complete *timelike* geodesic line.

It is also worth pointing out that the variational approach for Riemannian geodesics can be extended to the Lorentzian setting but important particularities appear. It is known since the old work by Uhlenbeck [113] that Morse theory can be applied to lightlike geodesics, under some conditions (including in particular global hyperbolicity). Lightlike geodesics satisfy also a relativistic *Fermat principle* [69, 94]. Combining both facts, one can study *gravitational lensing*, that is, the reception at some point p of the spacetime of light rays arriving in different directions from the same stellar object, the latter represented by some timelike curve c, see [92, 93]. As shown in [29], a very precise result on the existence and multiplicity of light rays from c to p in physically realistic spacetimes, can be stated in terms of the geodesics connecting two points for an appropriate Finsler metric. For the variational study of spacelike geodesics, see [81, 28] and references therein.

7 Mass, Penrose inequality and CCC

Asymptotically flat 4-spacetimes are useful to model the spacetime around an isolated body. They can be defined in terms of Penrose conformal embeddings, even though the definition is somewhat involved (see for example [114, 52]). Nevertheless, in what follows it is enough to bear in mind that, in an asymptotically flat (4-)spacetime there exists spacelike Cauchy hypersurfaces Σ which admits an *asymptotically flat* chart $(\Sigma\backslash K, (x_1, x_2, x_3))$ as follows. For some compact $K \subset \Sigma$ and some closed ball $\overline{B_0(R)}$ of $\mathbb{R}^3$, $\Sigma\backslash K$ is isometric to $\mathbb{R}^3\backslash\overline{B_0(R)}$ endowed with the metric:

$$h_{ij} - \delta_{ij} \in O(1/r), \quad \partial_k h_{ij} \in O(1/r^2), \quad \partial_k\partial_l h_{ij} \in O(1/r^3), \tag{4}$$

in Cartesian coordinates (this means that Σ is intrinsically asymptotically flat, as a Riemannian 3-manifold; in particular, Ric and S, are in $O(1/r^3)$), and, even more, its second fundamental form σ satisfies: $\sigma_{ij} \in O(1/r^2), \partial_k\sigma_{ij} \in O(1/r^3)$. (This definition can be extended to include more than one end, each one isometric to $(\Sigma\backslash K, (x_1, x_2, x_3))$ as above.)

The total ADM (Arnowit, Deser, Misner) *mass of an asymptotically flat Riemannian 3-manifold* can be defined as the limit in any asymptotic chart:

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \sum_{i,j=1}^{3} \int_{S_r} (\partial_i h_{ij} - \partial_j h_{ii})\, n^j dA, \tag{5}$$

where n is the outward unit vector to S_r, the sphere of radius r. Notice that m depends only on the Riemannian 3-manifold; in fact, when this manifold is seen as a hypersurface of an asymptotically flat spacetime, the appropriate name for m is *ADM energy*, and the definition of mass depends on σ, see the next section. This definition of mass is not mathematically elegant, but recall:

1. ADM mass appears naturally in a Hamiltonian approach, as an asymptotic boundary term for the variations of $\int S$. The definition is not trivial because

no strictly local notion of relativistic energy is available — nevertheless, it is worth pointing out the attempts to define a quasilocal mass [111].

2. There exists a classical Newtonian analog when the spacetime is Ricci-flat outside $\mathbb{R} \times K$, K compact, and there exists a timelike Killing vector field ξ with $\lim_{r\to\infty}|\xi| = 1$, such that $\Sigma \perp \xi$. In this case, the divergence theorem yields:
   $m = \frac{1}{4\pi}\int_K |\xi|^{-1}\mathrm{Ric}(\xi,\xi)dV = \int_K |\xi|\rho dV$
   i.e., the "integral of the poissonian density ρ measured at ∞".

3. The expression in coordinates for m is manageable:
   - If $h_{ij} = u^4\delta_{ij}$ with $u(x) = a + \frac{b}{|x|} + O(\frac{1}{|x|^2})$ then $m = 2ab$.
     In particular, this is the case if u is "harmonically flat" i.e. harmonic with finite limit at ∞.
   - Otherwise, when $S \geq 0$ then h is perturbable to the harmonically flat case with arbitrarily small error for m and preserving $S \geq 0$ (Schoen and Yau [106]; Corvino [41] extended the result for $m > 0$ without error in the mass).

4. Classical outer Schwarzschild metric can be written as:
   $M = \mathbb{R} \times \Sigma$, where $\Sigma = \mathbb{R}^3 \backslash \overline{B_0(|m|/2)}$;
   $g = -\left((1 - \frac{m}{2|x|})/u\right)^2 dt^2 + h$, $h_{ij} = u^4\delta_{ij}$ with $u = 1 + \frac{m}{2|x|}$
   (in particular $\sigma \equiv 0$). Of course, the classical Schwarzschild mass m agrees ADM mass.

One expects from the physical background that, when the dominant property holds, the ADM mass will be positive for any asymptotically flat Cauchy Σ. Two technical points are relevant here: (a) When Σ is totally geodesic ($\sigma \equiv 0$) the dominant property yields $S \geq 0$. (b) Under our definition of asymptotic flatness, Σ is necessarily complete, but the Riemannian part of exterior Schwarzschild spacetime $(\mathbb{R}^3 \backslash \overline{B_0(|m|/2)}, h)$ is incomplete for any $m \neq 0$. Of course, this is not a problem for the computation of the limit in the expression of the ADM mass, and one can also extend and modify $(\mathbb{R}^3 \backslash \overline{B_0(|m|/2)}, h)$ in a bounded region to obtain a complete Riemannian manifold Σ^c with the same asymptotic behaviour. Moreover, in the globally hyperbolic case $m > 0$, one can obtain such a Σ^c (say, corresponding to the spacetime created by a star of the same mass) with: (i) the same asymptotic behaviour, (ii) $S \geq 0$. Clearly, this property is not expected in the non-globally hyperbolic case $m < 0$. And, in fact, it is forbidden by the *Riemann positive mass theorem*:

Theorem 7.1 *Let* (Σ, h) *be any asymptotically flat (complete) Riemannian manifold with* $S \geq 0$. *Then,* $m \geq 0$ *and equality holds iff* (Σ, h) *is Euclidean space* $\mathbb{E}^3 = (\mathbb{R}^3, \delta)$.

Remark 7.2 This celebrated result by Schoen and Yau [105] is a purely Riemannian one. From this case, more general "positive mass" results follow, which include the case $\sigma \not\equiv 0$ [106]; see also the comments in the next section about the completely different Witten's proof. By the way, recall that the solution of Yamabe problem was completed by using the above result (see the nice survey [72]).

It is worth pointing out that, because of a technical problem which goes back to the known failure of regularity of minimal surfaces in dimensions greater than 7, the positive mass theorem is proved for dimensions up to 7 (this has been completed only recently, by using Schoen and Yau techniques, see [47]) as well as for spin manifolds of any dimension (by using Witten's techniques to be explained in the next section).

Next, we will consider a no less spectacular further step (for a detailed exposition, see [27]). But, first two notions will be briefly explained:

1.- *WCCC*. A question related with SCCC (see Remark 4.2) is the so-called *weak cosmic censorship conjecture* (WCCC), which is stated in the framework of asymptotically flat spacetimes. In such spacetimes, a natural notion of asymptotic future null infinity $\mathcal{J}^+$ can be defined ($\mathcal{J}^+$ is a subset of the image of M for a suitable conformal embedding in a bigger spacetime $\bar{M}$) and, then, also a rigorous notion of the *black hole* region B of M appears ($B = M \backslash J^-(\mathcal{J}^+)$) —this region corresponds to the intuitive idea of a "spatially bounded region from where nothing can escape". WCCC asserts that (maybe only generically) any spacetime M obtained as the maximal evolution of physically reasonable initial data with an asymptotic decay[5], will be asymptotically flat and, in a restrictive sense, globally hyperbolic at infinity[6]. The physical interpretation of this assertion is that no singularity (except at most an "initial" one) can be observed from $M \backslash B$, that is, singularities must lie inside a black hole and cannot be seen from outside (singularities are not "naked").

2.- *Outermost trapped surfaces.* Given a totally geodesic asymptotically flat slice Σ, those trapped surfaces (more precisely, compact spacelike surfaces whose expansion respect to the outer future lightlike direction is at no point positive) contained in Σ which are boundaries of a 3-manifold, are known to satisfy:

1. Such trapped surfaces correspond to compact minimal surfaces of Σ.
2. The outermost boundary compact minimal surfaces (necessarily topological spheres, each one the "apparent horizon in Σ of a black hole") are well-defined.
3. Let $\mathcal{H}$ be the union of the outermost minimal surfaces. Under WCCC, if $\mathcal{H}$ is connected and A_0 denotes its area, physical considerations ensure that the

[5]Typically, this data must satisfy: (i) (Σ, h, σ) is asymptotically flat, (ii) T satisfies the dominant property, and the equations for T constitute a quasilinear, diagonal, second order hyperbolic system, (iii) the fall-off of the initial value of T on Σ is fast enough for the h-distance, and h is assumed to be complete.

[6]More precisely, the latter means that the spacetime is *strongly asymptotically predictable*, see [114]. Recall that WCCC cannot be regarded as a particular case of SCCC.

"contribution to the mass" m_0 of the corresponding black hole would satisfy: $m_0 \geq \sqrt{\frac{A_0}{16\pi}}$.

Therefore, choosing any asymptotic Σ one expects for its mass m_Σ:

$$m_\Sigma \geq \sqrt{\frac{A_0}{16\pi}} \tag{6}$$

(at least if the second fundamental form vanishes). But (6) is an inequality in pure Riemannian Geometry. Thus, the following precise result must hold:

Theorem 7.3 *Let* (Σ, h) *be an asymptotically flat Riemannian 3-manifold with* $S \geq 0$*, and let* $\mathcal{H}_0$ *be the largest outermost (connected) minimal surface, with area* A_0*. Then inequality (6) is satisfied, and the equality holds if and only if* (Σ, h) *is Schwarzschild Riemannian metric outside* $\mathcal{H}_0$*.*

This is the celebrated "Riemann-Penrose inequality", proved by Huisken and Ilmanen [67] (who re-prove then the Riemann positive mass theorem), and shortly after extended by Bray to the full area of the (maybe non-connected) $\mathcal{H}$, with a different proof [26] based on positive mass theorem.

Penrose inequality is a more general conjecture, which includes the full spacetime case $\sigma \neq 0$ (recall that the case above would correspond when $(\Sigma, h, \sigma = 0)$ can be regarded as an initial data set for the spacetime). It is still open, and it becomes a major problem in Differential Geometry. An evidence of its difficulty is that it is supported by physical grounds, and counterexamples to more general appealing mathematical conjectures have been found, see [32]; we refer to the reviews [37, 78] for comprehensive references.

8 Spinors and holonomy

Dirac operators are popular objects of study in the area of global analysis, one of the main reasons being the existence of index theorems for them, see the standard textbook by Lawson and Michelson [71] or the book by Berline, Getzler, Vergne [20] on Dirac operators, both almost exclusively treating the Riemannian situation — this reflects the fact that index theory presently is applicable almost exclusively to elliptic operators. Another main reason of interest in spinors, more predominant in the Lorentzian case, is the presence of *Weitzenböck formulas*. These formulas reflect the fact that the Dirac operator as a natural first-order operator on spinors is a root of the Laplacian type operator plus a curvature-induced zeroth order term.

While in the book by Lawson-Michelson real spinors play a prominent role, in the following, we want to focus here on *complex* spinors.

Spinor bundles are defined verbatim in the same way as in the Riemannian case, with $SO(n)$ always replaced by the connected component of the identity of $SO(1, n)$, but there are some important differences of the Lorentzian to the Riemannian case: The natural (pseudo-Hermitean) scalar product $\langle \cdot, \cdot \rangle$ on the spinor bundle is not definite, but of split signature. Any timelike vector field X can be used to define a (non-natural) positive-definite scalar product $(\cdot,\cdot) := \langle X\cdot,\cdot\rangle$. Clifford multiplication is $\langle\rangle$-symmetric (instead of antisymmetric, as in the Riemannian case).

For a pseudo-Riemannian spin manifold of arbitrary signature, one can define the *Dirac operator* on C^1 (or at least $W^{1,p}$) sections ψ of the spinor bundle by $D\psi := \sum_{i=1}^{n} \epsilon_i e_i \cdot \nabla_{e_i}\psi$ (where the e_i are a pseudoorthogonal basis and ϵ_i is the sign of $g(e_i, e_i)$). The Dirac operator is formally self-adjoint, essentially self-adjoint if (M, g) is complete, and satisfies the Weitzenböck identity

$$D^2 = \nabla^*\nabla + \frac{1}{4}S,$$

where S is the pseudo-Riemannian scalar curvature of (M, g).

In the Riemannian situation, as the connection Laplacian $\nabla^*\nabla$ is positive-definite, the Weitzenböck formula is the initial point of many obstructions to positive scalar curvature for spin manifolds. The Weitzenböck formula for the Lorentzian Dirac operator looks superficially the same but the connection Laplacian here is a *hyperbolic* operator instead of an elliptic operator.

Exactly as in the Riemannian situation, any spinor defines an associated one-form, the so-called *Dirac current*. In Lorentzian geometry, an additional factor i appears in the definition, basically to balance the effect of the fore-mentioned differences. As in the Riemannian case, elementary calculations show that the Dirac current of a parallel spinor is a parallel spinor field. While in Riemannian geometry, the Dirac current of a *real* Killing spinor is a Killing vector field, in the Lorentzian case the same is true for *imaginary* Killing spinors. In stark contrast to the Riemannian situation, in Lorentzian geometry the Dirac current is always non-trivial for a non-vanishing vector field. The Dirac current of any eigenspinor of a twisted Dirac operator is always divergence-free, and $(\cdot,\cdot)$ can be used to define a conserved charge.

Many properties of spacetimes carrying special spinor fields can be read off from their Dirac current. E.g., as shown by Ehlers and Kundt [46], a four-dimensional Lorentzian spin manifold with a parallel spinor is locally isometric to a pp-wave (in this case, the Dirac current is a parellel null vector field).

An important application of spinors in Lorentzian geometry is the commented proof of the positive mass theorem due to the seminal ideas by Witten [115], made rigorous by Parker and Taubes [90] and others (see the independent work by Reula [97] as well as [66] and references therein). In fact, the spacetime viewpoint is necessary here. So, we will revisit the approach in the previous section, and focus in the positiveness of the energy (which, as commented above, could be also proven by using Schoen and Yau techniques). The starting point is a Cauchy hypersurface $(\Sigma, g|_\Sigma)$ that is asymptotically flat in the sense explained around formula (4), including the bounds for the second fundamental form. In this case, the expression of m in (5) is taken as the definition of the energy E, and the momenta P_l are defined by:

$$P_i = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \big(\sum_{j=1}^{3} \sigma_{ij} n^j - \sum_{j=1}^{3} \sigma_j^j n_i\big) dA,$$

where $\sum_{j=1}^{3} \sigma_j^j/3$ is the mean curvature of Σ. As we have seen, E is independent of the chosen asymptotic coordinates, and the freedom in the choice of these coordinates yields new momenta P_1', P_2', P_3' which differ in an element of $O(3)$. This allows to construct a vector $V := (E, P_1, P_2, P_3) \in \mathbb{R}^{1,3}$, the *ADM energy-momentum* (a different choice Σ' of hypersurface would yield a new vector V' which would be related to V). The statement of the positive energy theorem is that V is a causal vector and equal to 0 if and only if the spacetime is flat around Σ. Witten's idea was to use a Weitzenböck formula for the spacetime Dirac operator applied to spinors tangent to the hypersurface and extended parallely along normal geodesics in a small normal neighborhood of the Cauchy hypersurface. The dominant energy condition then ensures that the zeroth-order term in the Weitzenböck formula is positive, that yields directly the positiveness of the energy. Moreover, one can find a harmonic spinor approaching in coordinates a parallel spinor on $\mathbb{R}^3$ (thought of as in embedded via the Cauchy hypersurface) such that the limit of the boundary term appearing in the integral form of the Weitzenböck formula is exactly $E - |P|$, thus obtaining $E \geq |P|$ i.e., the energy momentum is a causal vector (a corollary is then the positive mass theorem $E \geq 0$). A central tool to prove existence of these solutions are Green's functions in weighted Sobolev spaces performed in detail by Parker and Taubes. In 1987, Yip [116] showed that the energy-momentum vector has to be even *timelike* (non-lightlike) unless M is flat around the Cauchy hypersurface, also by using spinors techniques. As already pointed out, Eichmair et al. [47] have given a recent proof of the energy-momentum inequality $E \geq |P|$ in the case that the manifold M is not necessarily spin, but that $\dim(M) \leq 7$. This is obtained by using Schoen and Yau techniques but, remarkably, a new difficulty appears, as minimal surfaces are now replaced by marginally outer trapped hypersurfaces, which do not come from a variational characterization.

Another fundamental concept in geometry connected to spinors is *holonomy*. That notion can be defined on any bundle with a connection, denoting, at a point p, the group of diffeomorphisms of the fiber over p which are parallel transports along curves starting and ending at p. If applied to a semi-Riemannian manifold of signature (m, n) with its Levi- Civita connection, it is a restriction of the standard representation of $SO(m, n)$ to a subgroup. It is easy to see that in a connected manifold, the equivalence class of the holonomy representation does not depend on the point p. The corresponding infinitesimal notion (taking the Lie algebra of the holonomy group) is called the *holonomy algebra*.

Now, in the Riemannian case, we have Berger's list: a simply-connected Riemannian manifold is either locally symmetric or can be decomposed as a Riemannian product each k-dimensional factor of which has the holonomy $SO(k)$, $U(k/2)$, $SU(k/2)$, $Sp(k/4) \cdot SP(1)$, $Sp(k/4)$, G_2 (in which case $k = 7$) or $Spin(7)$ (in which case $k = 8$). The Lorentzian case is a bit more involved and has remained open until recently. One of the difficulties compared to the Riemannian case is the difference between decomposability and reducibility pointed out in Section 2. In fact, classical de Rham Riemannian decomposition relies on the fact that the orthogonal complement $A^\perp$ of an invariant subspace A must be not only invariant too, but also a complement of A. When the latter property holds in the semi-Riemannian case, an analogous de Rham-Wu decomposition is obtained, but this is not the case when A is degenerate —a possibility that can hold in the Lorentzian case. So, the elementary building blocks of the Riemannian classification, irreducible subspaces, have therefore to be complemented by new, properly Lorentzian, building blocks, the indecomposable but non-irreducible subspaces. Such an m-dimensional subspace contains an invariant lightlike subspace N, and its holonomy algebra is contained in $(\mathbb{R} \oplus so(m-2)) \ltimes \mathbb{R}^{m-2}$, thus a central part of the classification is done by the $so(m)$-projections of the possible holonomy representation, the so-called *screen holonomy* acting on the associated invariant codimension-2 subbundle of τ_M, the *screen bundle* given by $N^\perp/N$ which carries a well-defined Riemannian metric. Thomas Leister in his PhD thesis in 2005 (published in [73]) showed that the screen holonomy is always the holonomy algebra of a Riemannian manifold (and, as remarked by Anton Galaev, this is an exceptional feature of Lorentzian geometry not present in higher signatures). In that manner, he was able to solve some of the remaining problems in the Lorentzian classification by means of the corresponding Riemannian techniques, and finally obtained the full classification. Galaev [54] gave then analytic examples for every holonomy representation of Leistner's list. Still, a missing piece were examples of *globally hyperbolic* manifolds *with complete Cauchy hypersurfaces* with the given holonomy representations. This was done for manifolds with parallel spinors (for which case the above classification yields groups $G \ltimes \mathbb{R}^n$ for G being a product of $SU(p), Sp(q), G_2$ or $Spin(7)$) in an article of Helga Baum and Olaf Müller [14], via a cylinder construction analogous to one by Bär, Gauduchon and Moroianu [8], building a parallel spinor from a so-called Codazzi spinor, and another construction relating Codazzi spinors to imaginary Killing spinors (whose importance in geometry is explained in the next paragraph). In 2013, Helga Baum and thomas Leistner then solved the analytic initial value problem for parallel spinors [16].

A *Killing spinor* is a spinor ψ such that there is a constant $b \in \mathbb{C}$ with $\nabla_X \psi = bX \cdot \psi$ for all vectors X. As shown by Friedrich in [53], Killing spinors can serve as landmarks where spectral estimates get sharp, in the following sense: If M is compact and the scalar curvature is bounded from below by a positive constant s_0, then for all eigenvalues a of the Dirac operator we have $a^2 \geq \frac{1}{4}\frac{n}{n-1}s_0$, and equality in this estimate implies that the corresponding eigenspinor is a Killing spinor. One can consider the modified connection $\nabla_b := \nabla - b\mathbf{1}$ to conclude that a Killing spinor never vanishes. An elementary calculation shows that $(\mathrm{Ric}(X) - 4b^2(n-1)X) \cdot \psi = 0$, and that implies (by nonvanishing of ψ) that the image $\mathrm{Ric} - 4b^2(n-1)\mathbf{1}_{TM}$ is contained in the null cone. Taking the trace once more, one sees that the scalar curvature equals $4n(n-1)b^2$, in particular, b is either real or purely imaginary. A Killing spinor is called real resp. imaginary if $b \in \mathbb{R}$ resp. $b \in i\mathbb{R}$. Over several years, different people aimed at a full classification of Killing spinors. Christian

Bär [7] finally came up with a cone construction which associated to each Killing spinor on a manifold M a parallel spinor on the Riemannian cone over M. As the existence of parallel spinors leads to special holonomy, Bär obtained a classification via the classification of Riemannian holonomies. Imaginary Killing spinors were classified by Helga Baum in [12], [13] in a completely different way: Let (M, g) be a complete connected spin manifold. It carries an $i \cdot a$-Killing spinor iff it is a warped product $\mathbb{R} \times_{e^{-4at}} N$ for a complete connected spin manifold with a non-zero parallel spinor field. The idea is to show that the manifold is foliated by level sets of the norm t of the spinor field. Christoph Bohle [25] examined real Killing spinors on Lorentzian manifolds, relating them also to warped products. Felipe Leitner [74], finally, considered imaginary Killing spinors on Lorentzian manifolds. Their Dirac current is easily seen to be causal, and when it is null, then the manifold is Einstein.

9 Some further topics and a double invitation

In this article, as announced, we intend to invite experts from other branches of mathematics, especially in Riemannian geometry and global analysis, in two respects: Firstly, we invite *users* of Lorentzian geometry. We hope that the article made clear that Lorentzian geometry can be extremely useful not only in physics, but also in mathematical contexts. One famous example is the aforementioned solution of the Riemannian Yamabe problem via Lorentzian techniques. To a large extent, this potential of Lorentzian geometry remains unexplored up to now.

Secondly, we want to invite *providers*. The open topics in Lorentzian geometry do need support from other branches of mathematics. In the following, we list some important open questions in Global Lorentzian Geometry (without any claim of completeness). In order to do so, it is convenient to distinguish between those that arise directly in Mathematical Relativity and those that are mathematically natural, independent of physical motivations. Along this paper we have emphasized some of the first type. But recall that the questions on Lorentzian manifolds inspired only in reasons of mathematical naturalness and beauty, are interesting in their own right and, sooner or later, will have applications to General Relativity or other parts of Mathematical Physics — recall that General Relativity is one of the two fundamental physical theories, and Quantum Theory the other one. For decades, the attempt to unify both theories has been a physical challenge and a permanent source of mathematical inspiration.

Along this article some open questions in Mathematical Relativity has appeared more or less explicitly, such as: (a) Cosmic Censorship Conjecture (weak and strong), including full Penrose inequality, (b) Cauchy problem (blow up criteria, global regularity for large data...), or (c) definitively satisfactory definition of singularities, including both singularity theorems (which involve divergences on curvature and not merely incompleteness) and a precise description of the *boundary* of the spacetime. Of course, there are many other relevant questions in Mathematical Relativity (see [37]). We would like to point out here the interest attracted by the questions of stability. Christodoulou and Klainerman [36] proved the non-linear stability of Lorentz-Minkowski spacetime $\mathbb{L}^4$ as a solution of Einstein equation. This means that a small perturbation of the initial conditions for $\mathbb{L}^4$ yields a spacetime with properties close to $\mathbb{L}^4$ (and, for example, not to a spacetime with singularities). In spite of the simplicity of this idea, the proof is extremely difficult -recall that [36] is a 500 pages book. The result is a landmark in Mathematical Relativity, and opens the study of the stability under weaker falloff hypotheses of the initial data or the stability of other spacetimes, as those with constant curvature, or of the Einstein equation coupled to other field theories.

Finally, let us point out some more purely mathematical questions, some of them suggested above, but only tangentially. (1) Classification of submanifolds with natural geometric properties (constant mean curvature, umbilic, etc.) in spaceforms and other physical or mathematically relevant spacetimes; notice that some of these questions had motivations from the viewpoint of the initial value problem and were commented in Section 5, but such problems evolve further, independent of physical motivations. (2) Critical curves for indefinite functionals on Lorentzian manifolds: even though the role of geodesics in General Relativity gives a general support for this, the infinite-dimensional variational mathematical approach for geodesics, including spacelike ones, has independent interest, see the seminal works by Benci, Fortunato and Giannoni [19], the book [81] of the review [28]; we emphasize that even a simple question as if any compact Lorentzian manifold must admit a closed geodesic remains open. (3) Curvature: curvature bounds groups have been stressed above, but there are many other questions related with curvature operators, e.g., those starting at the Osserman problem, solved a decade ago, see [58]. (4) Classification of Lorentzian spaceforms: such a topic has a deep importance and tradition in Geometry, we recommend the recent revision of a paper by Mess in [5] as an example of this exciting problem. (5) Links between Lorentzian and Finslerian geometries at different levels are being developed fast in the last years, see [30, 31, 51] as a sampler.

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