A note on invariant temporal functions

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Abstract

The purpose of this article is to present a result on the existence of Cauchy temporal functions invariant by the action of a compact group of conformal transformations in arbitrary globally hyperbolic manifolds. Moreover, the previous results about the existence of Cauchy temporal functions with additional properties on arbitrary globally hyperbolic manifolds are unified in a very general theorem. To make the article more accessible for non-experts, and in the lack of an appropriate single reference for the Lorentzian geometry background of the result, the latter is provided in an introductory section.

1 Introduction

From the viewpoint of a global analyst, the appropriate class of geometries to consider for symmetric hyperbolic systems are globally hyperbolic manifolds. These are defined by assuming two causality conditions. Historically, the row of the most important results in this context is certainly the construction of continuous Cauchy time functions due to Geroch [7] and of smooth Cauchy temporal functions due to Bernal and Sánchez [2], [3]. This latter statement ensures that in globally hyperbolic spacetimes, linear symmetric hyperbolic systems have well-defined initial value problems on Cauchy hypersurfaces. In a later work [8], Sánchez and the author proved a stronger statement ensuring even the existence of a *steep* Cauchy temporal function, thereby answering the question of an appropriate analogon of the Nash embedding theorem in the Lorentzian world. Now let us consider the initial value problem for Einstein theory, possibly coupled to any field theory. It is not an initial value problem of the sort mentioned above, as the metric is not a fixed background but a dynamical variable. Nevertheless, it is well-known by the work of Choquet-Bruhat and Geroch [6], using Zorn's lemma, that constrained initial data give rise to a unique maximal solution, and the question of global existence is replaced by the question of geodesic completeness of the maximal solution. If the initial conditions are invariant under a group G of diffeomorphisms (which are then isometries), the uniqueness statement implies that also the maximal solution is G-invariant. We can turn around the problem, begin with a G-invariant spacetime and ask if there are G-invariant initial conditions, that is, a G-invariant Cauchy surface and possibly even a G-invariant Cauchy temporal function. The author acknowledges Miguel Sánchez for the proposal of this question. The present note answers it in the affirmative for the case of a compact group G. At the same time, it combines the previous existence result for steep Cauchy temporal functions of [8] with the problem, solved in [4], of finding a Cauchy temporal function adapted to a Cauchy surface S, that is, a Cauchy temporal function taking the value zero on S. The main result, Theorem 1 below, states that we can require the three properties

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at once, steepness, adaptedness, and G-invariance for a compact group G that, if we do not care about steepness, does not even need to consist of isometries, but is only assumed to consist of conformal diffeomorphisms. Applying Theorem 1 in the case $m = 0$ yields that we do not have to assume a priori invariance of any acausal subset under G, except for the case that we look for a steep Cauchy temporal function adapted to some Cauchy surface S. Then, of course, it is necessary to assume G-invariance of S. If we want to adapt the Cauchy temporal function to more than one Cauchy surface, we cannot require steepness. Explicitly, the theorem reads:

Theorem 1 *Let, for $n \in \mathbb{N} \cup \{\infty\}$, (M, g) be a C^n Lorentzian manifold that is globally hyperbolic. Let $k \leq n$, $m \in \mathbb{N}$ and*[1] *let $(S^- = S_0, S_1, ..., S_m, S^+ = S_{m+1})$ be an $(m+2)$-tuple of C^k spacelike Cauchy hypersurfaces with $S_{i+1} \subset I^+(S_i)$ for all $0 \leq i \leq m+1$. Let $f_\pm : S^\pm \to \mathbb{R}$ be arbitrary continuous functions and let $a = (a_1, ...a_m)$ be an m-tuple of real numbers with $a_{i+1} > a_i$. Let G be a compact group of time-oriented conformal diffeomorphisms of (M, g). Let $t^\pm$ be future resp. past Cauchy time functions on $(I^+(S^+), g)$ resp. $(I^-(S^-), g)$. Then there is a C^{k-1} Cauchy temporal function T with*

(i) $S_i = T^{-1}(\{a_i\})$ *for all* $i \in \{1, ...m\}$,

(ii) $\pm T > \pm t^\pm/2 - 2$ *on* $I^\pm(S^\pm)$,

(iii) $\pm T|_{S^\pm} > f_\pm$.

If G leaves S_i invariant for all $1 \leq i \leq m$, then T can additionally been chosen G-invariant. If $m \in \{0, 1\}$ and if the group G consists of isometries, we can moreover find such a G-invariant T that is additionally steep.

The following proposition could easily be derived as a corollary of Theorem 1:

Proposition 2 *Let G be a compact group of conformal time-oriented diffeomorphisms of a globally hyperbolic spacetime. Then each G-orbit is acausal.*

However, we will give an independent proof of the proposition at the beginning of the third section already before entering the constructions.

The article is structured as follows:

The second section does not contain any new material. It is a short introduction into globally hyperbolic manifolds and continuous Cauchy time functions à la Geroch.

Aim of the third section is the proof of Theorem 1. The first proposition in this section, Prop. 13, and its proof is a slightly adapted version of the corresponding theorem and its proof in [8]. The theorem of this section comprises all previously published results on Cauchy temporal functions on globally hyperbolic manifolds, and even its two immediate corollaries, firstly the existence of a G-invariant smooth Cauchy temporal function and secondly of a steep Cauchy temporal function adapted to a given Cauchy surface, are new results.

2 Basics on Lorentzian manifolds and construction of continuous Cauchy time functions

The proofs of the results of this section can be found in [1], [5] and their references.

[1] Throughout this article, we use the convention that $0 \in \mathbb{N}$

Definition 3 *The* **signature** *of a nondegenerate symmetric bilinear form B on $\mathbb{R}^n$ is the tuple (r, s) where r is the number of negative and s is the number of positive eigenvalues of the endomorphism $i_h^{-1} \circ i_B$ where h is a positive definite bilinear form (here, for a bilinear form Z, by $i_Z : \mathbb{R}^n \to (\mathbb{R}^n)^*$ we denote the insertion into Z defined by $i_Z(v) := Z(v, \cdot)$). A vector $v \in \mathbb{R}^n$ is called* **spacelike** *resp.* **timelike** *resp.* **lightlike** *iff $B(v, v) > 0$ resp. $B(v, v) < 0$ resp. $B(v, v) = 0$.*

The nonspacelike nonzero vectors form a double cone C with tip 0, having two connected components. If we choose a timelike vector v, then one connected component, say C_1, has the property that each vector w in it satisfies $B(v, w) > 0$, whereas each vector u in the other component $C_2 = -C_1$ satisfies $B(v, u) < 0$.

Theorem 4 *(i) The inverse Cauchy-Schwarz inequality holds: For all nonspacelike $v, w \in \mathbb{R}^n$ we have $B(v, w)^2 \geq B(v, v) \cdot B(w, w)$*

(ii) The inverse triangle inequality holds: For every two nonspacelike vectors $v, w \in \mathbb{R}^n$ in the same connected component of nonspacelike vectors we have $\sqrt{-B(v + w, v + w)} \geq \sqrt{-B(v, v)} + \sqrt{-B(w, w)}$

(iii) $B(v + w, v + w) < B(v, v)$ for all v, w nonspacelike within the same connected component of C .

Definition 5 *Let M be a manifold of dimension n, let us denote the tangent bundle of M by $\tau M : TM \to M$. A* **Lorentzian metric on** *M is a section g of $\tau^* M \otimes \tau^* M$ (here starring denotes dualizing a bundle) such that g_p is of signature $(1, n-1)$ (via an arbitrary linear identification of T_pM with $\mathbb{R}^n$).*

We consider $I_g := \{v \in TM | g(v, v) < 0\}$, the subset of timelike vectors, and $J_g := \{v \in TM | g(v, v) \leq 0, v \neq 0\}$, the subset of causal vectors. For $p \in M$, we denote $J_{g,p} = J_g \cap T_pM$ and $I_{g,p} := I_g \cap T_pM$. Both $I_{g,p}$ and $J_{g,p}$ have two components.

Definition 6 *A Lorentzian manifold (M, g) is called* **time-orientable** *or* **spacetime** *iff J_g is disconnected.*

If (M, g) is time-oriented, then by standard arguments for sections into subsets of fiberwise nonempty interior, J_g has exactly two components. Every choice of a connected component J_g^+ is called **time-orientation of** (M, g) and J_g^+ is called future, $J_g^- := -J_g^+$ is called past, and $J_g = J_g^+ \cup J_g^-$. Then this notion of 'future' is transferred from TM to M, by means of curves.

Definition 7 *A C^0 and piecewise C^1 curve[2] $c : I \to M$, where I is a non-singleton interval, is called* **future-directed resp. past-directed causal curve** *or short* **future resp. past curve** *iff $\dot{c}(t) \in J_g^+$ resp. $\dot{c}(t) \in J_g^-$ for all $t \in I_j$ where the I_j are the differentiability intervals of c. The set of future curves from p to q is denoted by J_p^q and that of timelike future curves by I_p^q. Then we define $p \ll q$ iff $I_p^q \neq \emptyset$, $p \leq q$ iff either $p = q$ or $J_p^q \neq \emptyset$ and $J_g^\pm(p) := \{q \in M | p \leq q\}$ and $I_g^\pm(p) := \{q \in M | p \ll q\}$.*

Theorem 8 *Let (M, g) be a spacetime.*

(i) $I^+(A)$ is open for A an arbitrary subset.

(ii) **Push-up property:** *If $p \ll r \leq q$ or $p \leq r \ll q$, then $p \ll q$.*

[2]This regularity can be weakened, which is conceptually often more satisfying; for simplicity we nevertheless cling to piecewise C^1 curves in this introduction.

(iii) $J^+(p) \subset \overline{I^+(p)}$.

Let (M,g) be a spacetime. A subset $A \subset M$ is called **causally convex** if, for any causal curve C in M, $c^{-1}(A)$ is a connected interval (possibly empty or degenerate, i.e. a point). A subset A is called **achronal** resp. **acausal** iff there is no timelike curve between any two points of A, i.e. $I^+(p) \cap A = \emptyset$ for any $p \in A$, resp. if there is no future curve between any two points in A. A subset is called **future set** resp. **past set** if $A = I^+(A)$ resp. $A = I^-(A)$. For a subset $A \subset M$, the **past** resp. **future (causal) domain of dependence of** A is the set $D^-(A)$ resp $D^+(A)$ of points of M such that every C^0-inextendible[3] future resp. past curve meets A. The **domain of dependence of** A is $D(A) := D^-(A) \cup D^+(A)$. A subset A is called **Cauchy subset** if it is acausal and $D(A) = M$. In other words, A is a Cauchy subset if it is met exactly once by every C^0-inextendible future curve.

Definition 9 *A space-time (M,g) is called*

(i) **causal** *iff $J^+(p) \cap J^-(p) = \{p\}$ for all $p \in M$, i.e. iff there is no closed future curve in (M,g),*

(ii) **diamond-compact** *iff $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$,*

(iii) **globally hyperbolic** *iff it is causal and diamond-compact.*

Pseudo-Riemannian products $(\mathbb{R} \times N, -dt^2 + k)$ for (N,k) complete are globally hyperbolic, in particular Minkowski spaces are globally hyperbolic. Also causally convex subsets of globally hyperbolic manifolds are globally hyperbolic. Finally, iIf g, h are time-oriented Lorentz metrics on M with g globally hyperbolic and $J_h \subset J_g$, then h is globally hyperbolic as well. In particular a metric h conformally related to a globally hyperbolic metric g is globally hyperbolic. It is easy to see that for an acausal set A, the set $\mathrm{int}(D(A))$ is causally convex.
A function t is called **time function** iff $f \circ c$ is strictly increasing for every future curve c. If there is a time function on (M,g) then (M,g) is obviously causal. A function t is called **future-Cauchy** iff for every upper C^0-inextendible future curve $c : [0,b) \to M$, the set $t \circ c([0,b))$ is unbounded from above, and **past-Cauchy** iff for every upper C^0-inextendible past curve $c : [0,b) \to M$, the set $t \circ c([0,b))$ is unbounded from below. Here upper C^0-inextendibility means that one cannot extend the curve as a continuous map to $[0,b]$. A function is called **Cauchy** iff it is future-Cauchy and past-Cauchy, or equivalently, iff for every C^0-inextendible causal curve c, the map $t \circ c$ is surjective onto $\mathbb{R}$. If a time function is also Cauchy, then every of its level sets is a Cauchy subset.

Theorem 10 *Let (M,g) be a spacetime, let $k \in \mathbb{N} \cup \{\infty\}$ and let t be a C^k Cauchy time function on (M,g). Then (M,g) is C^k-diffeomorphic to $\mathbb{R} \times N$, where N is any level set of t.*

(M,g) is called **strongly causal** iff for every point $p \in M$ and any open neighborhood U of p there is an open causally convex neighborhood $W \subset U$ of p.

Lemma 11 *If (M,g) is diamond-compact and causal, then every point in M and every neighborhood U of p admits a geodesically convex and causally convex neighborhood $C \subset U$. In particular, (M,g) is strongly causal.*

Theorem 12 (Geroch) *Let (M,g) be globally hyperbolic, let μ be a volume form measure on M with $\mu(M) < \infty$, let the functions $t^\pm$ be defined by $t^\pm(x) := \mp\mu(J^\pm(x))$. Then $t := \ln(-t^-/t^+)$ is a continuous Cauchy time function.*

[3] If here and in the following we speak of C^0-inextendible future curves — meaning that after reparametrizing them to a bounded interval I there is no larger interval to which we could extend the curve as continuous map — the reader should nevertheless keep in mind that we always assume causal curves to be piecewise C^1.

3 Proof of Theorem 1

The aim of this section is to prove Theorem 1, that is, a G-invariant steep temporal function (possibly adapted to a given Cauchy surface) for a given compact group of conformal diffeomorphisms G. One necessary condition for this to hold is obviously that the orbits of G are acausal. This statement, formulated in Proposition 2 can be proven also directly:

Proof of Proposition 2. Assume that there are $p \in M$ and $gp \in J^+(p)$, then there is a future causal curve $c : [0,1] \to M$ from p to gp. Then, as the group G acts conformally and preserves the time orientation, the curve $k : [0,\infty) \to M$ defined by $k|_{[n,n+1]} = g^n \circ c$ for any $n \in \mathbb{N}$, is a piecewise C^1 future causal curve. Compactness of G implies that the subset $\{g^n | n \in \mathbb{N}\}$ has the identity, and thus also g, as accumulation points, thus k meets every neighborhood of p again after a prescribed time, in contradiction to strong causality. ∎

Let (M,g) be a spacetime and let A be a subset of M. A continuously differentiable function on M is called **temporal on** A iff it its gradient is past timelike on A, and **temporal** iff it is temporal on M. The interest in temporal functions becomes clear in the light of Theorem 10 as, if a temporal function t exists, it is automatically in the proof of Theorem 10, one can take $X = \mathrm{grad}(t)$ and gets not only a differential, but even a metric decomposition: (M,g) is then isometric to $(\mathbb{R} \times N, -f^2 dt^2 + \mathrm{pr}_2^* g_t)$ where $f \in C^\infty(M,(0,\infty))$ and $t \mapsto g_t$ is a smooth curve from $\mathbb{R}$ to the space of Riemannian metrics on N, equipped with the smooth compact-open topology, a Fréchet space topology on the space of symmetric bilinear forms in which the Riemannian metrics form an open cone.

Now, in order to improve this decomposition a bit, one can try to find a decomposition with f as above bounded. That leads immediately to the notion of *steep* Cauchy temporal functions. A continuously differentiable function t on M is called **steep on** A if there is some $c > 0$ such that $g(\mathrm{grad}(t), \mathrm{grad}(t)) < -c^2$ on A. It is called **steep** iff it is steep on (M,g). Obviously, every steep function is temporal. The first result of this section, for whose proof we slightly adapt the proof given in [8], is the existence of steep Cauchy temporal functions:

Proposition 13 (Existence of steep Cauchy temporal functions, compare with [8]) *Let (M,g) be globally hyperbolic and let t be a C^0 Cauchy time function on (M,g). Then (M,g) admits a steep Cauchy temporal function t_1 with $|t_1| > |t|/2 - 1$ and, thus, is isometric to a manifold $(\mathbb{R} \times N, -f^2 dt_1^2 + g_t)$ where $f \in C^\infty(M,(0,1])$ and $r \mapsto g_r$ is a smooth curve in the space of Riemannian metrics on N, such that all level sets of t_1 are Cauchy surfaces.*

So, in what follows (M,g) will be a globally hyperbolic spacetime, and we will assume that t is a Cauchy time function as given by Theorem 12. We also use the following notation:

$$T_a^b = t^{-1}([a,b]), \qquad S_a = t^{-1}(a).$$

For a Cauchy surface S, we say that $p \in M$ is S**-safe** iff there exists a geodesically convex neighborhood $U_p \subset V$ with $\partial^+ U_p \subset J^+(S)$, where $\partial^+ U_p := \partial U_p \cap J^+(p)$. In this case, let $j_p \in C^\infty(I^-(S))$ be the function

$$q \mapsto j_p(q) = \exp(-1/\eta(p,q)),$$

if $q \in I^+(p) \cap I^-(S)$ and 0 otherwise, where $\eta(p,q) := \langle v, v \rangle$ for $exp_p(v) = q$ if v is future timelike and 0 otherwise. For any two subsets $A, B \subset M$, we put

$$J(A, B) := J^+(A) \cap J^-(B)$$

and $J(p, B) := J(\{p\}, B)$.

Lemma 14 *Let τ be a function such that $g(\nabla\tau, \nabla\tau) < 0$ in some open subset U and let $K \subset U$ compact. For any function f there exists a constant c such that $g(\nabla(f + c\tau), \nabla(f + c\tau)) < -1$ on K.*

Proof. Notice that at each x in the compact subset K the quadratic polynomial $g(\nabla(f(x) + c\tau(x)), \nabla(f(x) + c\tau(x)))$ becomes smaller than -1 for some large c. ■

Lemma 15 *Let t be a Cauchy time function and let S be a level set of t, let $p \in J^-(S)$. For all neighborhood V of $J(p, S)$ there exists a smooth function $\tau \geq 0$ such that:*

(i) $\operatorname{supp} \tau \subset V$

(ii) $\tau > 1$ *on* $S \cap J^+(p)$.

(iii) $\nabla\tau$ *is timelike and past-directed in* $\mathrm{Int}(\mathrm{Supp}\ (\tau) \cap J^-(S))$.

(iv) $g(\nabla\tau, \nabla\tau) < -1$ *on* $J(p, S)$.

Proof. Let t take the value a on S, and let $K \subset V$ be a compact subset such that $J(p, S_a) \subset \mathrm{Int}\ (K)$. This compactness yields some $\delta > 0$ such that for every $x \in K$, x is $S_{t(x)+2\delta}$-safe.
Now, choose $a_0 < a_1 := t(p) < \cdots < a_n = a$ with $a_{i+1} - a_i < \delta/2$, and construct τ by induction on n as follows.
For $n = 1$, cover $J(p, S) = \{p\}$ with a set type $I^+(x) \cap U_x$ with $x \in K \cap T_{a_0}^{a_1}$ and consider the corresponding function j_x. For a suitable constant $c > 0$, the product cj_x satisfies both, (ii), (iii) and (iv). To obtain smoothability preserving (i), consider the open covering $\{I^-(S_{a+\delta}), I^+(S_{a+\delta/2})\}$ of M, and the first function $0 \leq \mu \leq 1$ of the associated partition of the unity ($\mathrm{Supp}\ \mu \subset I^-(S_{a+\delta})$). The required function is just $\tau = c\mu j_x$.
Now, assume by induction that the result follows for any chain $a_0 < \cdots < a_{n-1}$. So, for any $k \leq n-1$, consider $J(p, S_{a_k})$ and choose a compact set $\hat{K} \subset \mathrm{Int}\ K$ with $J(p, S) \subset \mathrm{Int}\ \hat{K}$. Then, there exists a function $\hat{\tau}$ which satisfies condition (i) above for $V = \mathrm{Int}\ \hat{K} \cap I^-(S_{a_{k+1}})$ and conditions (ii), (iii), (iv) for $S = S_{a_k}$. Now, cover $\hat{K} \cap T_{a_k}^{a_{k+1}}$ with a finite number of sets type $I^+(x^i) \cap U_{x^i}$ with $x^i \in K \cap T_{a_{k-1}}^{a_{k+1}}$, and consider the corresponding functions j_{x^i}.
For a suitable constant $c > 0$, the sum $\hat{\tau} + c \sum_i j_{x^i}$ satisfies (iii) for $S = S_{a_{k+1}}$. This is obvious in $J^-(S_{a_k})$ (for any $c > 0$), because of the convexity of timelike cones and the reversed triangle inequality. To realize that this can be also obtained in $T_{a_k}^{a_{k+1}}$, where $\nabla\tau$ may be non-timelike, notice that the support of $\nabla\hat{\tau}|_{T_{a_k}^{a_{k+1}}}$ is compact, and it is included in the interior of the support of $\sum_i j_{x^i}$, where the gradient of the sum is timelike, thus we can use Lemma 14. As $J^+(p, S_{a_{k+1}})$ is compact, conditions (ii), (iv) can be obtained by choosing, if necessary, a bigger c.
Finally, smoothability (and (i)), can be obtained again by using the open covering $\{I^-(S_{a_{k+1}+\delta}),\ I^+(S_{a_{k+1}+\delta/2})\}$ of M, and the corresponding first function μ of the associated partition of the unity, i.e. $\tau = \mu(\hat{\tau} + c \sum_i j_{x^i})$. ■

In order to extend locally defined temporal functions to a global temporal function, one cannot use a partition of the unity (as stressed in previous proof, as $\nabla\tau$ is not always timelike when μ is non-constant). Instead, local temporal functions must be added directly. To that purpose, adapted coverings are needed as those provided in the following definition:

Definition 16 *Let S be a Cauchy hypersurface. A* fat cone covering *of S is a sequence of points $p_i' \ll p_i, i \in \mathbb{N}$ such that both, $\mathcal{C}' = \{I^+(p_i') : i \in \mathbb{N}\}$ and $\mathcal{C} = \{I^+(p_i) : i \in \mathbb{N}\}$ yield a locally finite covering of S.*

Lemma 17 *Any Cauchy hypersurface S admits a fat cone covering $p_i' \ll p_i, i \in \mathbb{N}$. Moreover, both $\mathcal{C}$ and $\mathcal{C}'$ yield also a finite subcovering of $J^+(S)$.*

Proof. Let $\{K_j\}_j$ be a sequence of compact subsets of S satisfying $K_j \subset \text{Int } K_{j+1}$, $S = \cup_j K_j$. Each $K_j \backslash$ Int K_{j-1} can be covered by a finite number of sets type $I^+(p_{jk}), k = 1 \ldots k_j$ such that $I^+(p_{jk}) \cap S \subset K_{j+1} \backslash K_{j-2}$. Moreover, by continuity of the set-valued function I^+, this last inclusion is fulfilled if each p_{jk} is replaced by some close $p_{jk}' \ll p_{jk}$, and the required pairs $p_i'(= p_{jk}')$, $p_i(= p_{jk})$, are obtained. For the last assertion, take $q \in J^+(S)$ and any compact neighborhood $W \ni q$. As $J^-(W) \cap S$ is compact, it is intersected only by finitely many elements of $\mathcal{C}, \mathcal{C}'$, and the result follows. ∎

Definition 18 *Let $p', p \in T_{a-1}^a$, $p' \ll p$. A* steep forward cone function *for (a, p', p) is a smooth function $h^+_{a,p',p} : M \to [0, \infty)$ which satisfies the following:*

(i) $supp(h^+_{a,p',p}) \subset J^+(p', S_{a+2})$,

(ii) $h^+_{a,p',p} > 1$ on $S_{a+1} \cap J^+(p)$,

(iii) If $x \in J^-(S_{a+1})$ and $h^+_{a,p',p}(x) \neq 0$ then $\nabla h^+_{a,p',p}(x)$ is timelike and past-directed, and

(iv) $g(\nabla h^+_{a,p',p}, \nabla h^+_{a,p',p}) < -1$ on $J(p, S_{a+1})$.

Now, Lemma 15 applied to $S = S_{a+1}, V = I^-(S_{a+2}) \cap I^+(p')$ yields directly:

Lemma 19 *For all (a, p', p) there exists a steep forward cone function.* ∎

The existence of a fat cone covering (Proposition 17) allows to find a function h^a_+ which in some sense globalizes the properties of a steep forward cone function.

Lemma 20 *Choose $a \in \mathbb{R}$ and take any fat cone covering $\{p_i' \ll p_i | i \in \mathbb{N}\}$ for $S = S_a$. For every positive sequence $\{c_i \geq 1 | i \in \mathbb{N}\}$, the non-negative function $h^+_a := (|a| + 1) \sum_i c_i h^+_{a,p_i',p_i}$ satisfies:*

(i) $supp(h^+_a) \subset J(S_{a-1}, S_{a+2})$,

(ii) $h^+_a > |a| + 1$ on S_{a+1},

(iii) If $x \in J^-(S_{a+1})$ and $h^+_a(x) \neq 0$ then $\nabla h^+_a(x)$ is timelike and past-directed, and

(iv) $g(\nabla h^+_a, \nabla h^+_a) < -1$ on $J(S_a, S_{a+1})$.

Proof: Obvious. ∎

The gradient of h^+_a will be spacelike at some subset of $J(S_{a+1}, S_{a+2})$. So, in order to carry out the inductive process which proves Theorem 22, a strengthening of Lemma 20 will be needed.

Lemma 21 *Let $h_a^+ \geq 0$ as in Lemma 20. Then there exists a function h_{a+1}^+ which satisfies all the properties corresponding to Lemma 20 and additionally:*

$$g(\nabla(h_a^+ + h_{a+1}^+), \nabla(h_a^+ + h_{a+1}^+)) < -1 \qquad \text{on } J(S_{a+1}, S_{a+2}) \tag{1}$$

(so, this inequality holds automatically on all $J(S_a, S_{a+2})$).

Proof. Take a fat cone covering $\{p_i' \ll p_i | i \in \mathbb{N}\}$ for $S = S_{a+1}$. Now, for each p_i consider a constant $c_i \geq 1$ such that $c_i h^+_{a+1,p_i',p_i} + h_a^+$ satisfies inequality (1) on $J^+(p_i, S_{a+2})$ (see Lemma 14). The required function is then
$h_{a+1}^+ = (|a| + 2) \sum_i c_i h^+_{a+1,p_i',p_i}$. ∎

Now, we have the elements to complete the proof of Proposition 13.

Proof of Proposition 13. Consider the function h_a^+ provided by Lemma 20 for $a = 0$, and apply inductively Lemma 21 for $a = n \in \mathbb{N}$. Then, we obtain a function $t_1^+ = \sum_{n=0}^\infty h_n^+ \geq 0$ with nowhere spacelike gradient, which is a steep temporal function on $J^+(S_0)$ with support in $J^+(S_{-1})$. Analogously, one can obtain a function $t_1^- \geq 0$ which is a steep temporal function with the reversed time orientation, on $J^-(S_0)$. So, $t_1 := t_1^+ - t_1^-$ is clearly a steep temporal function on all M.
Moreover, the levels hypersurfaces of t_1 are Cauchy. In fact, consider any future-directed causal curve γ, and reparametrize it with the Cauchy time function t_0. Then, given a point x with $t(x) = a \in (0, \infty)$, by using the Gauß bracket $[\cdot]$ we obtain for all $x \in J^-(S_{-1}) \cup J^+(S_0)$:

$$t_1(x) \geq \inf t_1(t^{-1}([a])) = \inf t_1^+(t^{-1}([a])) \geq \inf h_n^+(t^{-1}([a])) > [a] + 1 \geq a = t(x)$$

and a corresponding statement for $a \in (-\infty, 0)$. Thus $|t_1| > |t|/2$ on $J^-(S_{-1}) \cup J^+(S_0)$.
In particular, any C^0-inextendible future curve γ crosses all the levels of t_1, as required. ∎

We call a C^1 function s on a spacetime (M, g) **almost-temporal** iff ∇s is zero or past everywhere.

Lemma 22 *Let (M, g) be globally hyperbolic, let t be a C^0 Cauchy time function on (M, g) and let S^- and S^+ be Cauchy surfaces of (M, g) with $S^+ \subset I^+(S^-)$. Let $f^\pm$ be a continuous positive function on $S^\pm$. Then there are steep Cauchy temporal functions $t_2^\pm(S^\pm, t)$ on (M, g) with:*

(i) $t_2^+(S^+, t) > t/2 - 1$ *on* $J^+(S^+)$ *and* $t_2^+(S^+, t)|_{S^+} > f^+$,

(ii) $t_2^-(S^-, t) < t/2 + 1$ *on* $J^-(S^-)$ *and* $t_2^-(S^-, t)|_{S^-} < f^-$.

Proof. First we choose an intermediate Cauchy surface S with $S^\pm \subset I^\pm(S)$. For the first assertion, take a steep Cauchy temporal function t_1 on (M, g) with $|t_1| > |t|/2 - 1$ on $J^-(S) \cup J^+(S^+)$ as in Proposition 13. Then choose a fat cone covering $\{(p_i', p_i)\}$ of S^+. For each i we choose a steep temporal function t^i on $I^+(p_i')$. Now, as $C_i := J^+(p_i) \cap S^+$ is a compact subset of $I^+(p_i')$, t^i takes a minimum m_i on C_i. Now take any smooth increasing function $\phi_i : \mathbb{R} \to \mathbb{R}$ with $\phi_i((-\infty, m_i - 1)) = \{0\}$ and $\phi_i'(x) = 1 \forall x \geq m_i$. Then $s_i := \phi_i \circ t^i$, extended by 0 on $M \setminus I^+(p_i')$ is a smooth nonnegative almost-temporal function which is steep on $I^+(C_i)$. By choosing an appropriate constant c_i one can ensure that $t_1 + cs_i > f$ on C_i. Now we consider $t_2^+(S, t) := t_1 + \sum_i c_i s_i$. Recall that the sum is a well-defined smooth function as the $I^+(p_i') \cap S$ have been chosen locally finite. And

$t_2(S^+, t) > t/2 - 1$ continues to hold on $J^+(\{t > 0\})$ for any choice of constants as we only add nonnegative terms (to show this, it is useful to distinguish the cases $t < 0$ and $t \geq 0$). For the second assertion, just time-dualize. ■

Lemma 23 *Let (M, g) be globally hyperbolic, let $S^- < S < S^+$ be C^k Cauchy hypersurfaces, let S be spacelike and of regularity C^k. Let $t^\pm$ be Cauchy time functions on $I^\pm(S)$. Then there is a C^{k-1} steep Cauchy temporal function t_3 with $t_3(S) = 0$, $\pm t_3|_{S^\pm} > f^\pm$ and $\pm t_3 > \pm t^\pm/2 - 2$ on $I^\pm(S)$.*

Proof. Let us first, along the lines of the corresponding proof in [4], construct a temporal function around S, which will then turn out to be steep on S.
Let $\nu : S \to TM$ be the future normal vector field of S and let W be a normal neighborhood of S, i.e., we assume that $E : S \times \mathbb{R} \supset A \to W$ is a diffeomorphism, where $E(s, r) := \exp(r \cdot \nu(s))$, and where $A \cap (\{s\} \times \mathbb{R})$ is a connected open interval containing 0. We define a causally convex subneighborhood $U \subset W$ of S in the following way: Choose a covering of S by subsets $I^-(p_i)$ for $p_i \in I^+(S)$ with $d(p_i, S) < 1$ and such that $U_i := I^-(p_i) \cap J^+(S) \subset W$. This can be constructed by considering Pseudo-Riemannian normal coordinates in a causally convex and geodesically convex neighborhood contained in W of a point $p \in S$. Analogously choose a covering of S by subsets $I^+(q_j)$ for $q_j \in I^-(S)$ such that $V_j := I^+(q_j) \cap J^-(S) \subset W$. Then the union U of $\bigcup_{i \in I} U_i$ and $\bigcup_{j \in J} V_j$ is easily seen to be causally convex and contained in W. Now we want to produce an almost-temporal function out of the signed distance function δ_S of S, which is C^{k-1} in W. To that aim, observe first that $U^\pm := U \cap I^\pm(S)$ is also causally convex, thus globally hyperbolic and therefore has a Cauchy surface S^{++} resp. S^{--} which is also a Cauchy surface for $I^\pm(S)$. Let $V := I^+(S^{--}) \cap I^-(S^{++})$.
Now we construct two almost-temporal functions $\theta^+ : M \to [0, 1]$ and $\theta^- : M \to [-1, 0]$ with the properties:

(i) $\theta^+(J^-(S^{--})) = \{0\}$, $\theta^+(J^+(S)) = \{1\}$,

(ii) $\theta^-(J^-(S)) = \{-1\}$, $\theta^-(J^+(S^{++})) = \{0\}$.

These functions can be defined easily by appropriate reparametrizations of Cauchy temporal functions on $V^\pm$. Then we put

$$\theta := 2\frac{(\delta_s + 1)\theta^+}{(\delta_s + 1)\theta^+ - \theta^-} - 1,$$

and we observe easily that θ is almost-temporal, $\theta(S) = 0$, θ can be smoothly extended to all of M by 1 on $J^+(S^{++})$ and by -1 on $J^-(S^{--})$. Moreover,

$$\nabla\theta|_S = \nabla\delta_s|_S,$$

thus θ is steep on S. As θ is a C^1 function, θ is steep in an open neighborhood V_0 of S. As each open neighborhood of S contains a causally convex subneighborhood of S (see above) we can find Cauchy surfaces $\tilde{S}^\pm \subset I^\pm(S) \cap V_0$. Now apply Lemma 22 to $(I^\pm(S), \tilde{S}^\pm)$ to get steep Cauchy temporal functions $t^\pm$ on $I^\pm(S)$ with $\pm t^\pm|_{\tilde{S}^\pm} > 1$. Then choose smooth functions $\phi^\pm : \rho \to \mathbb{R}$ with ϕ^+ increasing, $\phi^+(x) = 0$ for all $x < 0$, $\phi^+(x) \geq x$ for all $x \geq 1$ and $(\phi^+)'(x) \geq 1$ for all $x \geq 1$ and $\phi^-(y) = -\phi^+(-y)$ for all $y \in \mathbb{R}$. Then define functions $s^\pm$ on M by $s^+(x) := \phi^+(t^+(x))$ for all $x \in I^+(S)$ and $s^+(x) := 0$ for all $x \in J^-(S)$, as well as $s^-(x) := -\phi^-(-t^-(x))$ for all $x \in I^-(S)$ and $s^-(x) := 0$ for all $x \in J^+(S)$. Now, obviously, the function

$$\tilde{t}_3 := s^- + \theta + s^+$$

is steep on all of M, vanishes on S and satisfies the requirement involving $t^\pm$. To satisfy the requirement involving $f^\pm$ as well, we find another pair of smooth Cauchy temporal functions $T^\pm$ on $I^\pm(S)$ as in Lemma 22 with $T^+|_{S^+} > \max\{f^+, 1\}$ and $T^-|_{S^-} < \min\{f^-, -1\}$ and set $Z^\pm := \phi^\pm \circ T^\pm$ as in te definition of the $s^\pm$ above and

$$t_3 := Z^- + \tilde{t}_3 + Z^+,$$

which finally satisfies all our requirements. ∎

Note that, given an additional G-symmetry fixing S, one possible procedure in the proof above would have been to adapt U to the group action by setting $U := \bigcap_{g \in G} g(U)$. It can be shown that due to compactness of G this is still a nonempty open neighborhood of S, and it is still causally convex as intersection of causally convex subsets. However, we will not need this construction in the following.
Finally, we have all elements at hand for the proof of the main theorem that takes into account additionally a given compact group G of conformal diffeomorphisms:

Proof of Theorem 1. Let ω be a volume form on M. Then we can construct an invariant volume form ω^G on M by the well-known averaging process via the left-invariant Haar measure μ on G:

$$\omega^G := \int_G g^* \omega dg.$$

By convexity of $\Omega^n M$ it is easy to see that ω^G is indeed a smooth top form, and the normalization of μ entails that ω^G is indeed a volume form, which is, moreover, obviously left-invariant under G. Let S_1 be G-invariant and let $t^\pm$ be the Geroch time functions for the volume form ω^G. As G consists of conformal diffeomorphisms,

$$g(I^\pm(p)) = I^\pm(g(p)),$$

for all $p \in M$, and this, together with the G-invariance of S_1 and ω^G, implies that $t^\pm$ are G-invariant time functions on $I^\pm(S_1)$. Now let us first assume $m = 1$, and let $S := S_1$. Let t_3 be a smooth steep Cauchy temporal function with $t_3(S) = 0$ and $\pm t_3^\pm > t^\pm/2 \pm 2$ on $I^\pm(S^\pm)$. Existence of t_3 is ensured by Lemma 23. Then

$$t_4 := \int_G g^* t_3 d\mu(g)$$

is a G-invariant smooth function, well-defined because of compactness of G and the usual estimates. As S_A is G-invariant, it is adapted to S. If G consists of isometries, it is steep temporal because each map dg preserves $\{v \in TM | g(v, v) < a\}$ which is a fiberwise convex subset due to the inverse triangle inequality in T_pM, and because

$$dt_4(x) = \int_G d(g^* t_3) d\mu(g) = \int_G (dg \circ dt_3) d\mu(g),$$

as we can commute derivative and integral due to compactness of G. Finally, it is Cauchy as, given a number $r \in \mathbb{R}$, every C^0-inextendible future curve c has to reach $(t^+)^{-1}(2r)$, and $t_4(x) \geq r - 1$ for every $x \in (t^+)^{-1}(2r)$, a property inherited by the function t_3 (as all level sets of t^+ are G-invariant). The corresponding statement

holds for C^0-inextendible past curves. Thus $t_4 \circ c$ is surjective onto $\mathbb{R}$ for C^0-inextendible causal curves. The statement on a finite sequence of Cauchy surfaces, i.e. on the case $m > 1$, follows by an easy inductive argument:
Let a_i, S_i be given for all $i \in \{1, ..., n+1\}$. Then by inductional assumption there is $t^{[(1,\ldots,n)]}$ G-invariant with $t^{[(1,\ldots,n)]}(S_i) = \{a_i\}$ and $t^{[(1,\ldots,n)]}|_{S_{n+1}} > a_n$. We set

$$t^{[(1,\ldots,n+1)]} := \phi \circ t^{[(1,\ldots,n)]} + \psi \circ t^{[n+1]},$$

where $t^{[n+1]}$ is G-invariant, $t^{[n+1]}(S_{n+1}) = \{\frac{a_{n+1}-a_n}{2}\}$, $\psi, \phi : \mathbb{R} \to \mathbb{R}$ are monotonously increasing, $\phi(x) = x$ for all $x \leq a_n$, $\phi(x) = \frac{a_n+a_{n+1}}{2}$ for all $x \geq \frac{a_n+a_{n+1}}{2}$, and $\psi(x) = 0$ for all $x \leq 0$ and $\psi(x) = x$ for all $x \geq \frac{a_{n+1}-a_n}{2}$.

■

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