Local times for typical price paths and pathwise Tanaka formulas *

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Abstract

Following a hedging based approach to model free financial mathematics, we prove that it is possible to make an arbitrarily large profit by investing in those one-dimensional paths which do not possess local times. The local time is constructed from discrete approximations, and it is shown that it is of finite p-variation for all $p > 2$. Additionally, we provide various generalizations of Föllmer's pathwise Itô formula.

Key words: Itô formula, Local times, Model uncertainty, Tanaka formula.
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1 Introduction

In this paper, we use Vovk's [Vov12] game-theoretic approach to mathematical finance to construct local times for "typical price paths". Using these local times, we derive pathwise change of variable formulas in the spirit of Tanaka's formula. In particular, we can integrate $f(S)$ against a typical price path S for any f of finite q-variation for some $q < 2$.

This paper is a continuation of [PP13], where we used Vovk's approach to show that in a multidimensional setting every typical price path has a natural Itô rough path in the sense of Lyons [Lyo98] associated to it. Based on this, we set up a pathwise theory of integration which was motivated by possible applications in model free financial mathematics. We also showed that our integrals may be obtained as limit of Riemann sums, which allows for a nice financial interpretation as profit obtained by investing. With the techniques of [PP13] we are able to treat integrands that are not necessarily functions of the integrator. But if we want to construct $\int f(S)\,\mathrm{d}S$, then we need $f \in C^{1+\varepsilon}$. The aim of the current paper is to show that for one-dimensional processes this assumption can be greatly relaxed.

Our motivation comes amongst others from [DOR13], where pathwise local times and a pathwise generalized Itô formula are used to derive arbitrage free prices for weighted variance swaps in a model free setting. The techniques of [DOR13] allow to handle integrands in the

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Sobolev space H^1. Here we extend this to not necessarily continuous integrands of finite q-variation for some $q < 2$. Other motivations can be found in the survey paper [FS13] emphasizing possible applications of pathwise integration for robust hedging or in [CJ90], where local times appear naturally in a financial context and are used to resolve the so-called "stop-loss start-gain paradox".

We refer to [PP13] for a more detailed discussion of the need for pathwise stochastic integrals in model free finance.

Plan of the paper

In Section 2 we present various extensions of Föllmer's pathwise Itô formula under suitable assumptions on the local time. In Section 3 we show that typical price paths possess local times which satisfy all the assumptions of Section 2. Appendix A contains an alternative proof of our main result, based on Vovk's pathwise Dambis Dubins-Schwarz theorem.

2 Pathwise Tanaka formula

A first non-probabilistic approach to stochastic calculus was introduced by Föllmer in [Föl81], where an Itô formula was developed for a class of real-valued functions with quadratic variation. This builds our starting point for a pathwise version of Tanaka's formula and a generalized Itô formula, respectively. Let us start by recalling Föllmer's definition of quadratic variation.
A *partition* π is an increasing sequence $0 = t_0 < t_1 < ....$ without accumulation points, possibly taking the value ∞. For $T > 0$ we denote by $\pi[0,T] := \{t_j : t_j \in [0,T]\} \cup \{T\}$ the partition π restricted to $[0,T]$, and if $S\colon [0,\infty) \to \mathbb{R}$ is a continuous function we write

$$m(S,\pi[0,T]) := \max_{t_j\in\pi[0,T]} |S(t_j) - S(t_{j-1})|$$

for the mesh size of π along S on the interval $[0,T]$. We denote by $\mathcal{B}([0,\infty))$ the Borel σ-algebra on $[0,\infty)$.

Definition 2.1. Let (π^n) be a sequence of partitions and let $S \in C([0,\infty),\mathbb{R})$ be such that $\lim_{n\to\infty} m(S,\pi^n[0,T]) = 0$ for all $T > 0$. We say that S has *quadratic variation* along (π^n) if the sequence of measures

$$\mu_n := \sum_{t_j\in\pi^n} (S(t_{j+1}\wedge t) - S(t_j \wedge t))^2 \delta_{t_j}, \qquad n \in \mathbb{N},$$

on $([0,\infty),\mathcal{B}([0,\infty)))$ converges vaguely to a nonnegative Radon measure μ, where δ_t denotes the Dirac measure at $t \in [0,\infty)$. We write $\langle S\rangle_t := \mu([0,t])$ for the "distribution function" of μ and $\mathcal{Q}(\pi^n)$ for the set of all continuous functions having quadratic variation along (π^n).

We stress the fact that $\mathcal{Q}(\pi^n)$ depends on the sequence (π^n). For a given path S, the most natural sequence of partitions is maybe given by the dyadic Lebesgue partition generated by S: For each $n \in \mathbb{N}$ denote the set of dyadic points by $\mathbb{D}^n := \{k2^{-n} : k \in \mathbb{Z}\}$ and define the sequence

$$t_0^n := 0, \quad t_{k+1}^n := \inf\{t \geq t_k^n : S(t) \in \mathbb{D}^n \setminus S(t_k^n)\}, \quad k \in \mathbb{N}.$$

We set $\pi^n := \{0 = t_0^n < t_1^n < \dots\}$ and write $S \in \mathcal{Q}$ if $S \in \mathcal{Q}(\pi^n)$. Note that S may have constant parts, so that the mesh size of π^n does not necessarily converge to zero, but we always have $m(S,\pi^n[0,T]) \leq 2^{-n}$.

For $k \in \mathbb{N}$ let us write $C^k = C^k(\mathbb{R},\mathbb{R})$ for the space of k times continuously differentiable functions, and $C^k_b = C^k_b(\mathbb{R},\mathbb{R})$ for the space of functions in C^k that are bounded with bounded derivatives, equipped with the usual norm $\|\cdot\|_{C^k_b}$.

Föllmer provided a pathwise version of Itô's formula for $f \in C^2$ and $S \in \mathcal{Q}(\pi^n)$.

Theorem 2.2 ([Föl81]). *Let (π^n) be a sequence of partitions and let $S \in \mathcal{Q}(\pi^n)$ and $f \in C^2$. Then the pathwise Itô formula*

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\,\mathrm{d}S(s) + \frac{1}{2}\int_0^t f''(S(s))\,\mathrm{d}\langle S\rangle(s)$$

holds with

$$\int_0^t f'(S(s))\,\mathrm{d}S(s) := \lim_{n\to\infty}\sum_{t_j\in\pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j\wedge t)), \quad t \in [0,\infty), \tag{1}$$

where the series in (1) *is absolutely convergent. In particular, the integral $\int_0^\cdot g(S(s))\,\mathrm{d}S(s)$ is defined for all $g \in C^1$, and for all $T > 0$ the map $C^1_b \ni g \mapsto \int_0^\cdot g(S(s))\,\mathrm{d}S(s) \in C([0,T],\mathbb{R})$ defines a bounded linear operator and we have*

$$\Big|\int_0^t g(S_s)\,\mathrm{d}S_s\Big| \le |S_t - S_0| \times \|g\|_{L^\infty(\mathrm{supp}(S|_{[0,t]}))} + \frac{1}{2}\langle S\rangle_t \|g'\|_{L^\infty(\mathrm{supp}(S|_{[0,t]}))}$$

for all $t \ge 0$, where $\mathrm{supp}(S|_{[0,t]})$ *denotes the support of S restricted to the interval* $[0,t]$.

Föllmer actually requires the mesh size $\max_{t_j\in\pi^n, t_j\le T}|t_j - t_{j-1}|$ to converge to zero for all $T > 0$, but he also considers càdlàg functions S. For continuous S, the proof only uses that $m(S,\pi^n[0,T])$ converges to zero.

The continuity of the Itô integral is among its most important properties: if we approximate the integrand in a suitable topology, then the approximate integrals converge in probability to the correct limit. This is absolutely crucial in applications, for example when solving stochastic optimization problems or SDEs. Here we are arguing for one fixed path, so the statement in Theorem 2.2 is a natural formulation of the continuity properties in our context.

In the theory of continuous semimartingales, Itô's formula can be extended further to a generalized Itô rule for convex functions, see for instance Theorem 6.22 in [KS88]. In the spirit of Föllmer, a generalized Itô rule for functions in suitable Sobolev spaces was derived in the unpublished diploma thesis of Wuermli [Wue80]. We briefly recall here the idea for this pathwise version as presented in [Wue80] or [DOR13].

Let f' be right-continuous and of locally bounded variation, and set $f(x) := \int_{(0,x]} f'(y)\,\mathrm{d}y$ for $x \ge 0$ and $f(x) := -\int_{(x,0]} f'(y)\,\mathrm{d}y$ for $x < 0$. Then we get for $b \ge a$ that

$$f(b) - f(a) = f'(a)(b-a) + \int_{(a,b]} (f'(x) - f'(a))\,\mathrm{d}x = f'(a)(b-a) + \int_{(a,b]} (b-t)\,\mathrm{d}f'(t),$$

where we used integration by parts, and where the integral on the right hand side is to be understood in the Riemann-Stieltjes sense. For $b < a$, we get $f(b) - f(a) = f'(a)(b-a) + \int_{(b,a]}(t-b)\,\mathrm{d}f'(t)$. Therefore, for any $S \in C([0,\infty],\mathbb{R})$ and any partition π we have

$$\begin{aligned} f(S(t)) - f(S(0)) &= \sum_{t_j\in\pi} f'(S(t_j\wedge t))(S(t_{j+1}\wedge t) - S(t_j\wedge t)) \\ &\quad + \int_{-\infty}^{\infty} \sum_{t_j\in\pi} \Big(\mathbf{1}_{(\!(S(t_j\wedge t), S(t_{j+1}\wedge t)]\!]}(u)|S(t_{j+1}\wedge t) - u|\Big)\,\mathrm{d}f'(u), \end{aligned} \tag{2}$$

where we used the notation

$$(\!|u,v]\!] := \begin{cases} (u,v], & \text{if } u \le v, \\ (v,u], & \text{if } u > v, \end{cases}$$

for $u, v \in \mathbb{R}$. Let us define a discrete pathwise local time by setting

$$L_t^\pi(S,u) := \sum_{t_j \in \pi} \mathbf{1}_{(\!|S(t_j\wedge t),S(t_{j+1}\wedge t)]\!]}(u)|S(t_{j+1}\wedge t) - u|, \quad u \in \mathbb{R},$$

and note that $L_t^\pi(S,u) = 0$ for $u \notin [\inf_{s\in[0,t]} S(s), \sup_{s\in[0,t]} S(s)]$. In the following we may omit the S and just write $L_t^\pi(u)$.

Definition 2.3. Let (π^n) be a sequence of partitions and let $S \in C([0,\infty),\mathbb{R})$ be such that $\lim_{n\to\infty} m(S,\pi^n[0,T]) = 0$ for all $T > 0$. A function $L(S)\colon [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is called L^2-*local time* of S along (π^n) if for all $t \in [0,\infty)$ the discrete pathwise local times $L_t^{\pi^n}(S,\cdot)$ converge weakly in $L^2(\mathrm{d}u)$ to $L_t(S,\cdot)$ as $n \to \infty$. We write $\mathcal{L}_{L^2}(\pi^n)$ for the set of all continuous functions having an L^2-local time along (π^n). If (π^n) is the dyadic Lebesgue partition generated by $S \in \mathcal{L}_{L^2}(\pi^n)$, then we also write $S \in \mathcal{L}_{L^2}$.

Using this definition of the local time, Wuermli showed the following theorem, where we denote by $H^k = H^k(\mathbb{R},\mathbb{R})$ the Sobolev space of functions which are k times weakly differentiable in $L^2(\mathbb{R},\mathbb{R})$,

Theorem 2.4 ([Wue80], Satz 9 or [DOR13], Proposition B.4)**.** *Let (π^n) be a sequence of partitions and let $S \in \mathcal{L}_{L^2}(\pi^n)$. Then $S \in \mathcal{Q}(\pi^n)$, and for every $f \in H^2$ the generalized pathwise Itô formula*

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\,\mathrm{d}S(s) + \int_{-\infty}^\infty f''(u) L_t(S,u)\,\mathrm{d}u$$

holds with

$$\int_0^t f'(S(s))\,\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{t_j\in\pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j\wedge t)), \quad t \in [0,\infty).$$

(Note that f' is continuous for $f \in H^2$). In particular, the integral $\int_0^\cdot g(S(s))\,\mathrm{d}S(s)$ is defined for all $g \in H^1$, and for all $T > 0$, the map $H^1 \ni g \mapsto \int_0^\cdot g(S(s))\,\mathrm{d}S(s) \in C([0,T],\mathbb{R})$ defines a bounded linear operator. Moreover, for $A \in \mathcal{B}(\mathbb{R})$ we have the occupation density formula

$$\int_A L_t(u)\,\mathrm{d}u = \frac{1}{2}\int_0^t \mathbf{1}_A(S(s))\,\mathrm{d}\langle S\rangle(s), \quad t \in [0,\infty).$$

In other words, for all $t \ge 0$ the occupation measure of S on $[0,t]$ is absolutely continuous with respect to the Lebesgue measure, with density $2L_t$.

Sketch of proof. Formula (2) in combination with the continuity of f and S yields

$$\begin{aligned} f(S(t)) - f(S(0)) = &\sum_{t_j\in\pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j\wedge t)) \\ &+ \int_{-\infty}^\infty \sum_{t_j\in\pi^n} \Big(\mathbf{1}_{(\!|S(t_j\wedge t),S(t_{j+1}\wedge t)]\!]}(u)|S(t_{j+1}\wedge t) - u|\Big) f''(u)\,\mathrm{d}u. \end{aligned}$$

By assumption, the second term on the right hand side converges to $\int_{-\infty}^\infty f''(u)L_t(S,u)\,\mathrm{d}u$ as n tends to ∞, so that also the Riemann sums have to converge.

The occupation density formula follows by approximating $\mathbf{1}_A$ with continuous functions. □

As already observed by Bertoin [Ber87], the key point of this extension of Föllmer's pathwise stochastic integral is again that it is given by a *continuous* linear operator on H^1. Since $L_t(S)$ is compactly supported for all $t \geq 0$, the same arguments also work for functions f that are locally in H^2, i.e. such that $f|_{(a,b)} \in H^2((a,b),\mathbb{R})$ for all $-\infty < a < b < \infty$.

As we make stronger assumptions on the local times $L(S)$, it is natural to expect that we can extend Wuermli's generalized Itô formula to larger spaces of functions.

Definition 2.5. Let (π^n) be a sequence of partitions and let $S \in \mathcal{L}_{L^2}(\pi^n)$. We say that S has a *continuous local time* along (π^n) if for all $t \in [0,\infty)$ the discrete pathwise local times $L_t^{\pi^n}(S,\cdot)$ converge uniformly to a continuous limit $L_t(S,\cdot)$ as $n \to \infty$ and if $(t,u) \mapsto L_t(S,u)$ is jointly continuous. We write $\mathcal{L}_c(\pi^n)$ for the set of all S having a continuous local time along (π^n). If (π^n) is the dyadic Lebesgue partition generated by $S \in \mathcal{L}_c(\pi^n)$, then we also write $S \in \mathcal{L}_c$.

In the following theorem, $\mathrm{BV} = \mathrm{BV}(\mathbb{R},\mathbb{R})$ denotes the space of right-continuous bounded variation functions, equipped with the total variation norm.

Theorem 2.6. *Let (π^n) be a sequence of partitions and let $S \in \mathcal{L}_c(\pi^n)$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous with right-continuous Radon-Nikodym derivative f' of locally bounded variation. Then we have the generalized change of variable formula*

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(u))\,\mathrm{d}S(u) + \int_{-\infty}^{\infty} L_t(u)\,\mathrm{d}f'(u)$$

for all $t \in [0,\infty)$, where

$$\int_0^t f'(S(s))\,\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{t_j \in \pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j \wedge t)), \quad t \in [0,\infty).$$

In particular, the integral $\int_0^\cdot g(S_s)\,\mathrm{d}S_s$ is defined for all g of locally bounded variation, and for all $T > 0$ the map $\mathrm{BV} \ni g \mapsto \int_0^\cdot g(S(s))\,\mathrm{d}S(s) \in C([0,T],\mathbb{R})$ defines a bounded linear operator.

Proof. From (2) we get

$$f(S(t)) - f(S(0)) = \sum_{t_j \in \pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j \wedge t)) + \int_{-\infty}^{\infty} L_t^{\pi^n}(u)\,\mathrm{d}f'(u)\Big)$$

for all $t \geq 0$. Since $L_t^{\pi^n}$ converges uniformly to L_t, our claim immediately follows. □

Observe that f satisfies the assumptions of Theorem 2.6 if and only if it is the difference of two convex functions. As an immediate consequence of Theorem 2.6 we obtain a pathwise version of the classical Tanaka formula.

Corollary 2.7. *Let (π^n) be a sequence of partitions and let $S \in \mathcal{L}_c(\pi^n)$. The pathwise Tanaka-Meyer formula*

$$L_t(u) = (S(t) - u)^- - (S(0) - u)^- + \int_0^t \mathbf{1}_{(-\infty,u)}(S(s))\,\mathrm{d}S(s)$$

is valid for all $(t,u) \in [0,\infty) \times \mathbb{R}$, with the notation $(\cdot - u)^- := \max\{0, u - \cdot\}$. The analogous formulas for $\mathbf{1}_{[u,\infty)}(\cdot)$ and $\mathrm{sgn}(\cdot - u)$ hold as well.

At this point we see a picture emerge: the more regularity the local time has, the larger the space of functions is to which we can extend our pathwise stochastic integral. Indeed, the previous examples are all based on duality between the derivative of the integrand and the occupation measure. In the classical Föllmer-Itô case and for fixed time $T \geq 0$, the occupation measure is just a finite measure on a compact interval $[a, b]$, and certainly the continuous functions belong to the dual space of the finite measures on $[a, b]$. In the Wuermli setting, the occupation measure has a density in L^2 and therefore defines a bounded functional on L^2. If the local time is continuous, then we can even integrate Radon measures against it.

So if we can quantify the continuity of the local time, then the dual space further increases and we can extend the pathwise Itô formula to a bigger class of functions. To this end we introduce for a given sequence of partitions (π^n) and $p \geq 1$ the set $\mathcal{L}_{c,p}(\pi^n) \subseteq \mathcal{L}_c(\pi^n)$ consisting of those $S \in \mathcal{L}_c(\pi^n)$ for which the discrete local times $(L_t^{\pi^n})$ have uniformly bounded p-variation, uniformly in $t \in [0, T]$ for all $T > 0$, i.e. for which

$$\sup_{n\in\mathbb{N}} \|L^{\pi^n}\|_{C_T\mathcal{V}^p} := \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|L_t^{\pi^n}(\cdot)\|_{p-\mathrm{var}} < \infty$$

for all $T > 0$, where we write for any $f : \mathbb{R} \to \mathbb{R}$

$$\|f\|_{p-\mathrm{var}} := \sup\left\{ \left(\sum_{k=1}^{n} |f(u_k) - f(u_{k-1})|^p \right)^{1/p} : -\infty < u_0 < ... < u_n < \infty, n \in \mathbb{N} \right\}.$$

We also write $\mathcal{V}^p$ for the space of right-continuous functions of finite p-variation, equipped with the maximum of the p-variation seminorm and the supremum norm.
For $S \in \mathcal{L}_{c,p}(\pi^n)$ and using the Young integral it is possible to extend the pathwise Tanaka formula to an even larger class of integrands, allowing us to integrate $\int g(S)\,\mathrm{d}S$ provided that g has finite q-variation for some q with $1/p + 1/q > 1$. This is similar in spirit to the Bouleau-Yor [BY81] extension of the classical Tanaka formula. Such an extension was previously derived by Feng and Zhao [FZ06], Theorem 2.2. But Feng and Zhao stay in a semimartingale setting, and they interpret the stochastic integral appearing in (4) as a usual Itô integral. Here we obtain a pathwise integral, which is given very naturally as limit of Riemann sums.

Let us briefly recall the main concepts of Young integration. In [You36], Young showed that the integral $\int_0^t f\,\mathrm{d}g$ of two functions f and g is well defined as a limit of Riemann sums whenever f and g have finite p- respectively q-variation with $1/p + 1/q > 1$ and they have no common points of discontinuity. Of course, it is also defined whenever the Riemann-Stieltjes integral $\int_0^t f\,\mathrm{d}g$ exists, and in that case the two are equal. Moreover, there is $C(p, q) > 0$ such that whenever the Young integral exists, we have

$$\Big| \int_a^b f(s)\,\mathrm{d}g(s) \Big| \leq C(p, q) \|f\|_{p-\mathrm{var},[a,b]} \|g\|_{q-\mathrm{var},[a,b]} \tag{3}$$

for all $a < b$, where we wrote $\|f\|_{p-\mathrm{var},[a,b]} := \|f|_{[a,b]}\|_{p-\mathrm{var}}$ and similarly for g. We therefore easily obtain the following theorem.

Theorem 2.8 (see also [FZ06], Theorem 2.2)**.** *Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$. Let (π^n) be a sequence of partitions and let $S \in \mathcal{L}_{c,p}(\pi^n)$. Let $f: \mathbb{R} \to \mathbb{R}$ be absolutely continuous with right-continuous Radon-Nikodym derivative f' of locally finite q-variation. Then for all $t \in [0, \infty)$ the generalized change of variable formula*

$$f(S(t)) = f(S(0)) + \int_0^t f'(S(s))\,\mathrm{d}S(s) + \int_{-\infty}^{\infty} L_t(u)\,\mathrm{d}f'(u) \tag{4}$$

holds, where $\mathrm{d}f'(u)$ *denotes Young integration and where*

$$\int_0^t f'(S(s))\,\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{t_j\in\pi^n} f'(S(t_j))(S(t_{j+1}\wedge t) - S(t_j\wedge t)), \quad t\in[0,\infty).$$

In particular, the integral $\int_0^\cdot g(S_s)\,\mathrm{d}S_s$ *is defined for all right-continuous* g *of locally finite* q*-variation, and for all* $T>0$ *the map* $\mathcal{V}^q \ni g \mapsto \int_0^\cdot g(S(s))\,\mathrm{d}S(s) \in C([0,T],\mathbb{R})$ *defines a bounded linear operator.*

Proof. Observe that for finite n, formula (2) holds for $L_t^{\pi^n}$ and f', because while f' might no longer be of bounded variation, the discrete local time $L_t^{\pi^n}$ is. Since furthermore $L_t^{\pi^n}$ is piecewise smooth, there are no problems with possible common discontinuities of $L_t^{\pi^n}$ and f', and the integral

$$\int_{-\infty}^{\infty} L_t^{\pi^n}(u)\,\mathrm{d}f'(u)$$

is given as the limit of Riemann sums. In other words, the Young integral $\int_{-\infty}^{\infty} L_t^{\pi^n}(u)\,\mathrm{d}f'(u)$ exists. But then it must satisfy the bound (3). Since the p-variation of $(L_t^{\pi^n})$ is uniformly bounded, and the sequence converges uniformly to L_t, it is easy to see that it must converge in p'-variation for all $p'<p$. Choosing such a p' with $1/q+1/p'>1$ and combining the linearity of the Young integral with the bound (3), the result follows. □

Remark 2.9. *Theorem 2.2 in [FZ06] states* (4) *under the slightly weaker assumption that* $f\colon\mathbb{R}\to\mathbb{R}$ *is left-continuous and locally bounded with left-continuous and locally bounded left derivative* D^-f*. But absolute continuity of* f *is clearly necessary: Consider the path* $S(t)\equiv t$ *for* $t\in[0,\infty)$*, for which* $\langle S\rangle\equiv 0$ *and thus* $L\equiv 0$*. In this case equation* (4) *would read as*

$$f(t) = f(0) + \int_0^t \mathrm{D}^-f(u)\,\mathrm{d}u, \quad t\in[0,\infty),$$

a contradiction if f *is not absolutely continuous.*

In the following, we will show that any typical price path which might model an asset price trajectory must be in $\mathcal{L}_{c,p}$.

3 Local times for model free finance

3.1 Super-hedging and outer measure

In a recent series of papers [Vov11a, Vov11b, Vov12], Vovk introduced a hedging based, model free approach to mathematical finance. Roughly speaking, Vovk considers the set of real-valued continuous functions as price paths and introduces an outer measure on this set which is given by the cheapest super-hedging price. A property (P) is said to hold for "typical price paths" if it is possible to make an arbitrarily large profit by investing in the paths where (P) is violated. We will see that in Vovk's framework it is possible to construct continuous local times for typical price paths, which leads to an axiomatic justification for the use of our pathwise generalized Itô formulas from Section 2 in model free finance.

More precisely, we consider the (sample) space $\Omega = C([0,\infty),\mathbb{R})$ of all continuous functions $\omega\colon[0,\infty)\to\mathbb{R}$. The coordinate process on Ω is denoted by $S_t(\omega) := \omega(t)$. For $t\in[0,\infty)$ we define $\mathcal{F}_t := \sigma(S_s : s\le t)$ and we set $\mathcal{F} := \bigvee_{t\ge 0}\mathcal{F}_t$. Stopping times τ and the associated σ-algebras $\mathcal{F}_\tau$ are defined as usual.

A process $H\colon \Omega \times [0,\infty) \to \mathbb{R}$ is called a *simple strategy* if there exist stopping times $0 = \tau_0(\omega) < \tau_1(\omega) < \dots$ such that for every $\omega \in \Omega$ we have $\tau_n(\omega) = \infty$ for all but finitely many n, and $\mathcal{F}_{\tau_n}$-measurable bounded functions $F_n\colon \Omega \to \mathbb{R}$ such that $H_t(\omega) = \sum_{n\geq 0} F_n(\omega)\mathbf{1}_{(\tau_n(\omega),\tau_{n+1}(\omega)]}(t)$. In that case the integral

$$(H \cdot S)_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega)[S_{\tau_{n+1}(\omega)\wedge t} - S_{\tau_n(\omega)\wedge t}]$$

is well defined for every $\omega \in \Omega$ and every $t \in [0,\infty)$. Moreover, $(H \cdot S)$ is asymptotically constant so that $(H^n \cdot S)_\infty(\omega) := \lim_{t\to\infty}(H \cdot S)_t(\omega)$ exists for all $\omega \in \Omega$.

For $\lambda > 0$ a simple strategy H is called λ-*admissible* if $(H \cdot S)_t(\omega) \geq -\lambda$ for all $t \in [0,\infty)$ and all $\omega \in \Omega$. The set of λ-admissible simple strategies is denoted by $\mathcal{H}_\lambda$.

Definition 3.1. The *outer measure* of $A \subseteq \Omega$ is defined as the cheapest superhedging price,

$$\overline{P}(A) := \inf\Big\{\lambda > 0 : \exists\, (H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_\infty(\omega)) \geq \mathbf{1}_A(\omega)\,\forall\omega \in \Omega\Big\}.$$

Of course, it would be more natural to minimize over simple trading strategies rather than over the limit inferior along sequences of simple strategies. But then $\overline{P}$ would not be countably subadditive, and this would make it very difficult to work with. Let us just remark that in the classical definition of superhedging prices in semimartingale models we work with general admissible strategies, and the Itô integral against a general strategy is given as limit of integrals against simple strategies. So in that sense our definition is analogous to the classical one.

A set of paths $A \subseteq \Omega$ is called a *null set* if it has outer measure zero. A property (P) holds for *typical price paths* if the set A where (P) is violated is a null set.

The most important property of $\overline{P}$ is the following arbitrage interpretation for null sets.

Lemma 3.2 (Lemma 4 of [PP13]). *A set $A \subseteq \Omega$ is a null set if and only if there exists a sequence of 1-admissible simple strategies $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_1$, such that*

$$\liminf_{n\to\infty}(1 + (H^n \cdot S)_\infty) \geq \infty \cdot \mathbf{1}_A(\omega),$$

where we set $\infty \cdot 0 = 0$.

In other words, a null set is essentially a model free arbitrage opportunity of the first kind, and to only work with typical price paths is analogous to only considering models which satisfy (NA1) (no arbitrage opportunities of the first kind). The notion (NA1) has raised a lot of interest in recent years since it is the minimal conditions which has to be satisfied by any reasonable asset price model; see for example [KK07, Ruf13, IP11]. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, we say that S satisfies *(NA1)* under $\mathbb{P}$ if the set $\mathcal{W}_1^\infty := \{1 + \int_0^\infty H_s \,\mathrm{d}S_s : H \in \mathcal{H}_1\}$ is bounded in probability, that is if $\lim_{n\to\infty} \sup_{X\in\mathcal{W}_1^\infty} \mathbb{P}(X \geq n) = 0$.

In the next proposition we collect further properties of $\overline{P}$. For proofs (in finite time) see [PP13].

Proposition 3.3. *1. $\overline{P}$ is an outer measure with $\overline{P}(\Omega) = 1$, i.e. $\overline{P}$ is nondecreasing, countably subadditive, and $\overline{P}(\emptyset) = 0$.*

2. An equivalent definition of $\overline{P}$ is

$$\overline{P}(A) = \inf\Big\{\lambda > 0 : \exists\, (H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda \textit{ s.t. } \liminf_{n\to\infty} \sup_{t\in[0,\infty)} (\lambda + (H^n \cdot S)_t(\omega)) \geq \mathbf{1}_A(\omega)\,\forall\omega \in \Omega\Big\}, \quad A \subseteq \Omega.$$

3. *Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that the coordinate process S is a $\mathbb{P}$-local martingale, and let $A \in \mathcal{F}$. Then $\mathbb{P}(A) \leq \overline{P}(A)$.*

4. *Let $A \in \mathcal{F}$ be a null set, and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that the coordinate process S satisfies (NA1) under $\mathbb{P}$. Then $\mathbb{P}(A) = 0$.*

The last statement says that every property which is satisfied by typical price paths holds quasi-surely for all probability measures which might be of interest in mathematical finance.

Lemma 3.2 and Proposition 3.3 are originally due to Vovk, but here and in [PP13] we consider a small modification of Vovk's outer measure, which in our opinion has a slightly more natural financial interpretation and with which it is easier to work.

3.2 Existence of local times for typical price paths

We are now ready to prove our main result: every typical price path has a local time which satisfies all the requirements needed to apply our most general Itô-Tanaka formula, Theorem 2.8.

Lemma 3.4. *Let $\pi = \{0 = t_0 < t_1 < ... < t_K < t_{K+1} = \infty\}$ be a partition of $[0, \infty)$ and let $S \in C([0, \infty), \mathbb{R})$. Then we have for all $t \in [0, \infty)$ and all $u \in \mathbb{R}$*

$$L_t^\pi(S, u) = (S_t - u)^- - (S_0 - u)^- + \sum_{j=0}^{K} \mathbf{1}_{(-\infty,u)}(S_{t_j})[S_{t_{j+1}\wedge t} - S_{t_j\wedge t}], \tag{5}$$

where we recall that $L_t^\pi(S, u) = \sum_{j=0}^K \mathbf{1}_{(\!(S_{t_j\wedge t}, S_{t_{j+1}\wedge t}]\!]}(u)|S_{t_{j+1}\wedge t} - u|$.

Proof. This is a special case of (2), but we can also give a direct and elementary proof: For all $j = 1, ..., K$, notice that

$$\begin{aligned}\mathbf{1}_{(\!(S_{t_j\wedge t}, S_{t_{j+1}\wedge t}]\!]}(u)|S_{t_{j+1}\wedge t} - u| &= \mathbf{1}_{(\!(S_{t_j\wedge t}, S_{t_{j+1}\wedge t}]\!]}(u)\big((S_{t_{j+1}\wedge t} - u)^+ + (S_{t_{j+1}} - u)^-\big) \\ &= \mathbf{1}_{(-\infty,u)}(S_{t_j\wedge t})(S_{t_{j+1}\wedge t} - u)^+ + \big(1 - \mathbf{1}_{(-\infty,u)}(S_{t_j\wedge t})\big)(S_{t_{j+1}\wedge t} - u)^- \\ &= \mathbf{1}_{(-\infty,u)}(S_{t_j\wedge t})(S_{t_{j+1}\wedge t} - u) + (S_{t_{j+1}\wedge t} - u)^-,\end{aligned}$$

which leads to

$$\begin{aligned}L_t^\pi(S, u) &= \sum_{j=0}^{K} \big[\mathbf{1}_{(-\infty,u)}(S_{t_j\wedge t})(S_{t_{j+1}\wedge t} - u) + (S_{t_{j+1}\wedge t} - u)^-\big] \\ &= (S_{t_K\wedge t} - u)^- - (S_{t_0\wedge t} - u)^- + \sum_{j=0}^{K} \mathbf{1}_{(-\infty,u)}(S_{t_j\wedge t})\big[(S_{t_{j+1}\wedge t} - u) + (S_{t_j\wedge t} - u)^-\big] \\ &= (S_t - u)^- - (S_0 - u)^- + \sum_{j=0}^{K} \mathbf{1}_{(-\infty,u)}(S_{t_j})[S_{t_{j+1}\wedge t} - S_{t_j\wedge t}],\end{aligned}$$

for all $(t, u) \in [0, \infty) \times \mathbb{R}$. ☐

Therefore, the construction of the stochastic integral $\int_0^\cdot \mathbf{1}_{(-\infty,u)}(S_s)\,\mathrm{d}S_s$ is equivalent to the construction of the local time $L_\cdot(S, u)$.

Let us introduce the dyadic Lebesgue partition generated by S: For each $n \in \mathbb{N}$ denote the set of dyadic points by $\mathbb{D}^n := \{k2^{-n} : k \in \mathbb{Z}\}$ and define the sequence of stopping times

$$\tau_0^n(\omega) := 0, \quad \tau_{k+1}^n(\omega) := \inf\{t \geq \tau_k^n(\omega) : S_t(\omega) \in \mathbb{D}^n \setminus S_{\tau_k^n(\omega)}(\omega)\}, \quad k \in \mathbb{N}. \tag{6}$$

We set $\pi^n(\omega) := \{0 = \tau_0^n(\omega) < \tau_1^n(\omega) < \dots\}$. Note that the functions τ_k^n are stopping times and that $(\pi^n(\omega))$ is *increasing*, i.e. for all $n, k \in \mathbb{N}$ there exists k' such that $\tau_k^n(\omega) = \tau_{k'}^{n+1}(\omega)$. In the following we omit the ω and just write π^n and τ_k^n instead of $\pi^n(\omega)$ and $\tau_k^n(\omega)$.

A key ingredient for our construction of the local time is the following analysis of the number of interval crossings. Let $U_t(\omega, a, b)$ be the number of upcrossings of the closed interval $[a, b] \subseteq \mathbb{R}$ by $\omega \in \Omega$ during the time interval $[0, t]$, and write $D_t(\omega, a, b)$ for the number of downcrossings.

Lemma 3.5. *For typical price paths $\omega \in \Omega$, there exists $C(\omega)\colon (0, \infty) \to (0, \infty)$ such that*

$$\max_{k\in\mathbb{Z}} \big(U_T^n(\omega, k2^{-n}) + D_T^n(\omega, k2^{-n})\big) \leq C_T(\omega) n^2 2^n$$

for all $n \in \mathbb{N}$, $T > 0$, where $U_T^n(\omega, u) := U_T(\omega, u, u + 2^{-n})$ for $u \in \mathbb{R}$, and similarly for the downcrossings.

Proof. Let $K, T > 0$. Without loss of generality we may restrict our considerations to the set $A_K := \{\omega \in \Omega : \sup_{t\in[0,T]} |S_t(\omega)| < K\}$. Let $k \in (-2^n K, 2^n K)$ and write $u = k2^{-n}$. The following strategy will make a large profit if $U_T^n(u)$ is large: start with wealth 1, when first hitting u invest $1/(2K)$ into S. When S hits $-K$ sell and stop trading. Otherwise, when S hits $u + 2^{-n}$ sell. This gives us wealth $1 + 2^{-n}/(2K)$ on the set $\{U_T^n(u) \geq 1\} \cap A_K$. Now we repeat this strategy: next time we hit u, we invest our current wealth times $1/(2K)$ into S, and sell when S hits $u + 2^{-n}$. After $n^2 2^n$ upcrossings of $[u, u + 2^{-n}]$, stop trading. On the set $\{U_T^n(u) \geq n^2 2^n\} \cap A_K$ we then have a wealth of

$$\left(1 + \frac{2^{-n}}{2K}\right)^{n^2 2^n} \geq \exp\left(\frac{1}{4K} n^2\right)$$

for all n that are large enough. Therefore

$$\bar{P}\big(\{U_T^n(u) \geq n^2 2^n\} \cap A_K\big) \leq \exp\left(-\frac{n^2}{4K}\right)$$

for all large n. Summing over all dyadic points $u = k2^{-n}$ in $(-K, K)$, we obtain

$$\overline{P}\Big(\Big\{\max_{k\in\mathbb{Z}} U_T^n(k2^{-n}) \geq n^2 2^n\Big\} \cap A_K\Big) \leq K 2^{n+1} \exp\Big(-\frac{n^2}{4K}\Big) = K \exp\Big(-\frac{n^2}{8K} + (n+1)\log(2)\Big)$$

for all large n. Since this is summable in n, the claimed bound for the upcrossings follows for all typical price paths. To bound the downcrossings, it suffices to note that up- and downcrossings of a given interval differ by at most 1. □

The following construction is partly inspired by [MP10], Chapter 6.2.

Theorem 3.6. *Let $T > 0$, $\alpha \in (0, 1/2)$ and (π^n) as defined in (6). For typical price paths $\omega \in \Omega$, the discrete local time L^{π^n} converges uniformly in $(t, u) \in [0, T] \times \mathbb{R}$ to a limit $L \in C([0, T], C^\alpha(\mathbb{R}))$, and there exists $C = C(\omega) > 0$ such that*

$$||L^{\pi^n} - L||_{L^\infty([0,T]\times\mathbb{R})} \leq C 2^{-n\alpha}. \tag{7}$$

Moreover, for all $p > 2$ we have $\sup_{n\in\mathbb{N}} ||L^{\pi^n}||_{C_T \mathcal{V}^p} < \infty$ for typical price paths.

Proof. By Lemma 3.4 it suffices to prove the corresponding statements with the stochastic integrals $\int_0^t 1_{(-\infty,u)}(S_s)\,\mathrm{d}S_s$ replacing $L_t(S,u)$. Using Lemma 3.5, we may fix $K>0$ and restrict our attention to the set

$$A_K := \left\{\omega\in\Omega : \sup_{t\in[0,T]}|S_t(\omega)|<K \text{ and } \max_{k\in\mathbb{Z}}\left(U_T^n(\omega,k2^{-n})+D_T(\omega,k2^{-n})\right)\le Kn^2 2^n \text{ for all } n\right\}.$$

Let $u\in(-K,K)$. For every $n\in\mathbb{N}$ we approximate $\mathbf{1}_{(-\infty,u)}(S_\cdot)$ by the process

$$F_t^n(u) := \sum_{k=0}^{\infty}\mathbf{1}_{(-\infty,u)}(S_{\tau_k^n})\mathbf{1}_{[\tau_k^n,\tau_{k+1}^n)}(t),\quad t\ge 0.$$

Then we write for the corresponding integral process

$$I_t^{\pi^n}(u) := \sum_{k=0}^{\infty}\mathbf{1}_{(-\infty,u)}(S_{\tau_k^n}(\omega))[S_{\tau_{k+1}^n\wedge t}(\omega)-S_{\tau_k^n\wedge t}(\omega)],\quad t\ge 0,$$

and since (π^n) is increasing, we get

$$I_t^{\pi^n}(u)-I_t^{\pi^{n-1}}(u)=\sum_{k=0}^{\infty}[F_{\tau_k^n}^n(u)-F_{\tau_k^n}^{n-1}(u)][S_{\tau_{k+1}^n\wedge t}-S_{\tau_k^n\wedge t}].$$

By the construction of our stopping times (τ_k^n), we have

$$\sup_{t\ge 0}\left|[F_{\tau_k^n}^n(u)-F_{\tau_k^n}^{n-1}(u)][S_{\tau_{k+1}^n\wedge t}(\omega)-S_{\tau_k^n\wedge t}(\omega)]\right|\le 2^{-n+2}.$$

Hence, the pathwise Hoeffding inequality, Theorem 3 in [Vov12] or Lemma 35 in [PP13], implies for every $\lambda\in\mathbb{R}$ the existence of a 1-admissible simple strategy $H^\lambda\in\mathcal{H}_1$, such that

$$1+(H^\lambda\cdot S)_t(\omega)\ge\exp\left(\lambda(I_t^{\pi^n}(u)-I_t^{\pi^{n-1}}(u))-\frac{\lambda^2}{2}N_t^n(u,\omega)2^{-2n+4}\right)=:\mathcal{E}_t^{\lambda,n}(\omega)$$

for all $t\in[0,T]$ and all $\omega\in\Omega$, where $N_t^n(u):=N_t^n(u,\omega)$ denotes the number of stopping times $\tau_k^n\le t$ with $F_{\tau_k^n}^n(u)-F_{\tau_k^n}^{n-1}(u)\ne 0$. Now observe that F_t^n and F_t^{n-1} are constant on dyadic intervals of length 2^{-n}, which means that we may suppose without loss of generality that $u=k2^{-n}$ is a dyadic number. But we can estimate $N_T^n(k2^{-n})$ by the number of upcrossings of the interval $[(k-1)2^{-n},k2^{-n}]$ plus the number of the downcrossings of the interval $[k2^{-n},(k+1)2^{-n}]$, which means that on A_K we have $N_T^n(u)\le 2K2^n n^2$. So considering $(H^\lambda+H^{-\lambda})/2$ for $\lambda>0$, we get

$$\overline{P}\left(\left\{\sup_{t\in[0,T]}|I_t^{\pi^n}(u)-I_t^{\pi^{n-1}}(u)|\ge 2^{-n\alpha}\right\}\cap A_K\right)\le 2\exp(-\lambda 2^{-n\alpha}+\lambda^2 K2^{-n+4}n^2)$$

for all $\lambda,\alpha>0$. Choose now $\lambda=2^{n/2}$ and $\alpha\in(0,1/2)$. Then we get the estimate

$$\overline{P}\left(\left\{\sup_{t\in[0,T]}|I_t^{\pi^n}(u)-I_t^{\pi^{n-1}}(u)|\ge 2^{-n\alpha}\right\}\cap A_K\right)\le 2\exp(-2^{n(1/2-\alpha)}+16Kn^2).$$

Moreover, noting that for all $t>0$ the maps $u\mapsto I_t^{\pi^n}(u)$ and $u\mapsto I_t^{\pi^{n-1}}(u)$ are constant on dyadic intervals of length 2^{-n} and that there are $2K2^n$ such intervals in $[-K,K]$, we can simply estimate

$$\begin{aligned}\overline{P}\left(\left\{\sup_{(t,u)\in[0,T]\times\mathbb{R}}|I_t^{\pi^n}(u)-I_t^{\pi^{n-1}}(u)|\ge 2^{-n\alpha}\right\}\cap A_K\right)&\\ &\le 2K2^n\times 2\exp(-2^{n(1/2-\alpha)}+16Kn^2)\\ &=\exp(-2^{n(1/2-\alpha)}+16Kn^2+(n+2)\log 2+\log K).\end{aligned}$$

Obviously, this is summable in n and thus the proof of the uniform convergence and of the speed of convergence is complete.

It remains to prove the uniform bound on the p-variation norm of I^{π^n} and the Hölder continuity of the limit. Let $p > 2$ and write $\alpha = 1/p$, so that $\alpha \in (0, 1/2)$. First let $u = k2^{-n} \in (-K, K)$ and write $v = (k+1)2^{-n}$. Then

$$I_t^{\pi^n}(v) - I_t^{\pi^n}(u) = \sum_{k=0}^{\infty} (F_{\tau_k^n}^n(v) - F_{\tau_k^n}^n(u))(S_{\tau_k^n \wedge t} - S_{\tau_{k-1}^n \wedge t}),$$

and similarly as before we have $\sup_{t\geq 0} |(F_{\tau_k^n}^n(v) - F_{\tau_k^n}^n(u))(S_{\tau_k^n \wedge t} - S_{\tau_{k-1}^n \wedge t})| \leq 2^{-n+1}$. On A_K, the number of stopping times $(\tau_k^n)_k$ with $F_{\tau_k^n}^n(u) \neq F_{\tau_k^n}^n(v)$ is bounded from above by $2K2^n n^2 + 1$, and therefore we can estimate as before

$$\overline{P}\left(\left\{ \sup_{t\in[0,T]} \sup_{u,v\in\mathbb{R}:|u-v|\leq 2^{-n}} |I_t^{\pi^n}(v) - I_t^{\pi^n}(u)| \geq 2^{-n\alpha} \right\} \cap A_K \right) \leq \exp(-2^{n(1/2-\alpha)} + Cn^2),$$

for some appropriate constant $C = C(K) > 0$.

We conclude that for typical price paths $\omega \in \Omega$ there exists $C = C(\omega) > 0$ such that

$$\sup_{t\in[0,T]} \sup_{|u-v|\leq 2^{-n}} |I_t^{\pi^n}(v) - I_t^{\pi^n}(u)| + \sup_{t\in[0,T]} \sup_{u\in\mathbb{R}} |I_t^{\pi^n}(u) - I_t^{\pi^{n-1}}(u)| \leq C2^{-n\alpha}$$

for all $n \in \mathbb{N}$. Let now $n \in \mathbb{N}$ and let $u, v \in \mathbb{R}$ with $1 \geq |u - v| \geq 2^{-n}$. Let $m \leq n$ be such that $2^{-m-1} < |u - v| \leq 2^{-m}$. Then

$$\begin{aligned} ||I^{\pi^n}(v) - I^{\pi^n}(u)||_\infty &\leq ||I^{\pi^n}(v) - I^{\pi^m}(v)||_\infty + ||I^{\pi^m}(v) - I^{\pi^m}(u)||_\infty + ||I^{\pi^m}(u) - I^{\pi^n}(u)||_\infty \\ &\leq C \left(\sum_{k=m+1}^{n} 2^{-k\alpha} + 2^{-m\alpha} + \sum_{k=m+1}^{n} 2^{-k\alpha} \right) \\ &\leq C2^{-m\alpha} \leq C|v-u|^\alpha, \end{aligned}$$

possibly adapting the value of $C > 0$ in every step. Since $I_t^{\pi^n}$ is constant on dyadic intervals of length 2^{-n}, this proves that $\sup_{t\in[0,T]} ||I_t^{\pi^n}||_{p-var} \leq C$. The α-Hölder continuity of the limit is shown in the same way. □

We reduced the problem of constructing L to the problem of constructing certain integrals. In [PP13], Theorem 16, we gave a general pathwise construction of stochastic integrals. But this result does not apply here, because in general $\mathbf{1}_{(-\infty,u)}(S)$ is not càdlàg.

Remark 3.7. *Theorem 3.6 gives a simple, model free proof that local times exist and have nice properties. Let us stress again that by Proposition 3.3, all the statements of Theorem 3.6 hold quasi-surely for all probability measures on $(\Omega, \mathcal{F})$ under which S satisfies (NA1).*

Below, we sketch an alternative proof based on Vovk's pathwise Dambis Dubins-Schwarz theorem. While we are interested in a statement for typical price paths, which a priori is stronger than a quasi sure result for all measures satisfying (NA1), the quasi-sure statement may also be obtained by observing that every process satisfying (NA1) admits a dominating local martingale measure, see [Ruf13, IP11]. Under the local martingale measure we can then perfom a time change to turn the coordinate process into a Brownian motion, and then we can invoke standard results for Brownian motion for which all statements of Theorem 3.6 except one are well known: The only result we could not find in the literature is the uniform boundedness in p-variation of the discrete local times.

Remark 3.8. *Note that for $u = k2^{-n}$ with $k \in \mathbb{Z}$ we have $L_t^{\pi^n}(u) = 2^{-n} D_t(u - 2^{-n}, u) + \varepsilon(n,t,u)$ for some $\varepsilon(n,t,u) \in [0, 2^{-n}]$. Therefore our proof also shows that the renormalized downcrossings converge uniformly to the local time, with speed at least $2^{-n\alpha}$ for $\alpha < 1/2$. Of course, for the Brownian motion this is well known, see [CLPT81]; see also [Kho94] for the exact speed of convergence. In the Brownian case, we actually know more: Outside of one fixed null set we have*

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}} \sup_{t \in [0,T]} |\varepsilon^{-1} D_t(x, x+\varepsilon) - L_t(x)| = 0$$

for all $T > 0$. It should be possible to recover this result also in our setting. It follows from simple analytic estimates once we prove Theorem 3.6 for a sequence of partitions $(\widetilde{\pi}^n)$ of the following type: Let (c_n) be a sequence of strictly positive numbers converging to 0, such that c_{n+1}/c_n converges to 1. Define $\widetilde{\mathbb{D}}^n := \{kc_n : k \in \mathbb{Z}\}$. Now define $\widetilde{\pi}^n$ as π^n, replacing $\mathbb{D}^n$ by $\widetilde{\mathbb{D}}^n$. The only problem is that then we cannot expect the sequence $(\widetilde{\pi}^n)$ to be increasing, and this would complicate the presentation, which is why we prefer to work with the dyadic Lebesgue partition.

A Existence of local times via time change

A remarkable result in [Vov12] is a pathwise Dambis Dubins-Schwarz theorem, which allows to link results for the one-dimensional Wiener process to typical price paths. This opens another way of showing the existence of local times, which we will briefly sketch here.

For that purpose let us recall Vovk's outer measure and relate it to ours. For $\lambda \in (0, \infty)$ we define the set of processes

$$\mathcal{S}_\lambda := \left\{ \sum_{k=0}^{\infty} H^k : H^k \in \mathcal{H}_{\lambda_k}, \lambda_k > 0, \sum_{k=0}^{\infty} \lambda_k = \lambda \right\}.$$

For every $G = \sum_{k \geq 0} H^k \in \mathcal{S}_\lambda$, all $\omega \in \Omega$, and all $t \in [0, \infty)$, the integral

$$(G \cdot S)_t(\omega) := \sum_{k \geq 0} (H^k \cdot S)_t(\omega) = \sum_{k \geq 0} (\lambda_k + (H^k \cdot S)_t(\omega)) - \lambda$$

is well defined and takes values in $[-\lambda, \infty]$. Vovk then defines

$$\overline{Q}(A) := \inf \left\{ \lambda > 0 : \exists\, G \in \mathcal{S}_\lambda \text{ s.t. } \lambda + \liminf_{t \to \infty} (G \cdot S)_t(\omega) \geq \mathbf{1}_A(\omega) \,\forall \omega \in \Omega \right\}, \quad A \subseteq \Omega.$$

It is fairly easy to show that $\overline{P}(A) \leq \overline{Q}(A)$ for all $A \subseteq \Omega$, see Section 2.1 of [PP13]. In other words, all results which hold true outside of a $\overline{Q}$-null set are also true outside of a $\overline{P}$-null set.

To state Vovk's pathwise Dambis Dubins-Schwarz theorem, we need to define time-superinvariant sets.

Definition A.1. A continuous non-decreasing function $f\colon [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ is said to be a *time change*. A subset $A \subseteq \Omega$ is called *time-superinvariant* if for each $\omega \in \Omega$ and each time change f it is true that $\omega \circ f \in A$ implies $\omega \in A$.

For $x \in \mathbb{R}$ we denote by μ_x the Wiener measure on $(\Omega, \mathcal{F})$ with $\mu_x(\omega(0) = x) = 1$.

Lemma A.2. *For every time-superinvariant set $A \subseteq \Omega$ satisfying $\omega(0) = x$ for all $\omega \in A$ and $\mu_x(A) = 0$, we have $\overline{P}(A) = 0$.*

Proof. Using Theorem 1 in [Vov12], we obtain $\overline{P}(A) \leq \overline{Q}(A) = \mu_x(A) = 0$. □

First we investigate in the next lemma the behavior of local times under a time change. Recall that $\mathcal{L}_c$ is the set of those paths S which are in $\mathcal{L}_c(\pi^n)$ for the dyadic Lebesgue partition (π^n) constructed from S.

Lemma A.3. *Let $S \in \mathcal{Q}$ and assume that for all $t \geq 0$ the occupation measure*

$$\mu_t(A) = \int_0^t \mathbf{1}_A(S(s))\,\mathrm{d}\langle S\rangle(s), \quad A \in \mathcal{B}(\mathbb{R}),$$

is absolutely continuous with density $2L_t(S)$. Let f be a time change. Then $S \circ f \in \mathcal{Q}$ and the occupation measure of $S \circ f$ is absolutely continuous with density $2L_{f(t)}(S)$ for all $t \geq 0$.

Proof. Recall that $\langle S\rangle$ is constructed along the dyadic Lebesgue partition, which yields $\langle S \circ f\rangle_t = \langle S\rangle_{f(t)}(\omega)$. The result then follows by considering the push forward of the occupation measure of S under f. □

With the previous lemma at hand we can reduce the existence and continuity of local times for typical price paths to the case of the Wiener process. For $p \geq 1$ let us define the events

$$A_c := \{\omega \in \Omega : S(\omega) \in \mathcal{L}_c\} \quad \text{and}$$
$$A_{c,p} := \{\omega \in A_c : u \mapsto L_t(S(\omega), u) \text{ has finite } p\text{-variation for all } t \in [0,\infty)\}.$$

Theorem A.4. *Typical price paths are in $A_{c,p}$ for all $p > 2$.*

Proof. Define $\Omega_x := \{\omega \in \Omega : \omega(0) = x\}$ for $x \in \mathbb{R}$. Lemma A.2 and Lemma A.3 in combination with classical results for the Wiener process (see [KS88], Theorem 3.6.11 or [MP10], Theorem 6.19) show that typical price paths $\omega \in \Omega_x$ have an absolutely continuous occupation measure with jointly continuous density $\{2L_t(S,u)\,;\,(t,u) \in [0,\infty) \times \mathbb{R}\}$. In [MP10], Theorem 6.19 it is also shown that $u \mapsto L_t(S,u)$ has finite p-variation, uniformly bounded in $t \in [0,T]$, for all $T > 0$, $p > 2$. It remains to show the uniform convergence of the discrete local times to L and to get rid of the restriction $\omega \in \Omega_x$.

Recall that $U_t(S,a,b)$ and $D_t(S,a,b)$ denote the number of up- respectively downcrossings of the interval $[a,b]$ completed by S up to time t. First observe that

$$\left|L_t^{\pi^n}(S,u) - 2^{-n} D_t(S, u - 2^{-n}, u)\right| \leq 2^{-n} \tag{8}$$

for all $t \in [0,\infty)$ and $u \in \mathbb{D}^n$. For $u \in \mathbb{R}$ we define $\{u\}_n := \min\{k \in \mathbb{D}^n : k \geq u\}$ and by the triangle inequality we read

$$\begin{aligned}
&\sup_{(t,u)\in[0,T]\times\mathbb{R}} \left|L_t^{\pi^n}(S,u) - L_t(S,u)\right| \\
&\quad\leq \sup_{(t,u)\in[0,T]\times\mathbb{R}} \left|L_t^{\pi^n}(S,u) - L_t^{\pi^n}(S,\{u\}_n)\right| + \sup_{(t,u)\in[0,T]\times\mathbb{R}} \left|L_t^{\pi^n}(S,\{u\}_n) - L_t(S,\{u\}_n)\right| \\
&\qquad + \sup_{(t,u)\in[0,T]\times\mathbb{R}} \left|L_t(S,\{u\}_n) - L_t(S,u)\right|.
\end{aligned}$$

Now we separately deal with the three summands. The discrete Tanaka formula (5) yields

$$\left|L_t^{\pi^n}(S,u) - L_t^{\pi^n}(S,\{u\}_n)\right| \leq 3 \cdot 2^{-n}$$

for all $(t,u) \in [0,T] \times \mathbb{R}$.
For the second summand we remark that the event

$$E := \left\{\omega \in \Omega_x : \limsup_{n\to\infty} \sup_{(t,u)\in[0,T]\times\mathbb{R}} \left|2^{-n} D_t(S, u - 2^{-n}, u) - L_t(S,u)\right| > 0 \text{ for some } T \in [0,\infty)\right\}$$

is time-superinvariant. Therefore, it suffices to combine Theorem 2 in [CLPT81] with (8) to obtain that the second summand converges to zero for typical price paths.
That the last summand goes to zero simply follows from the joint continuity of the occupation density $L(S)$ in (t, u).

Finally, we indicate how to get rid of the assumption $\omega \in \Omega_x$ for some $x \in \mathbb{R}$. For $\varepsilon > 0$ it suffices to fix a sequence of simple trading strategies $(H^n) \subset \mathcal{H}_\varepsilon$ with

$$\liminf_{n\to\infty}(\varepsilon + (H^n \cdot S)_T(\omega)) \geq 1$$

for all $\omega \in \Omega_0$ for which the local time does not exist. Applying these simple trading strategies to $\omega - \omega(0)$ achieves the same aim but without the restriction $\omega(0) = 0$. □

Remark A.5. *1. For Theorem A.4, the dyadic points $\mathbb{D}^n$ in the definition of (π^n) can be replaced by any increasing sequence of partitions $(\mathcal{P}^n)$ of $\mathbb{R}$ such that $\lim_{n\to\infty} |\mathcal{P}^n| = 0$; see [CLPT81].*

2. While Theorem A.4 gives us the uniform convergence to a jointly continuous local time which is of finite p-variation in u, it does not give us the uniform boundedness in p-variation of the approximating sequence (L^{π^n}). Therefore, the results of Theorem A.4 only allow us to prove an abstract version of Theorem 2.8, where the pathwise stochastic integral $\int_0^t g(S(s))\,\mathrm{d}S(s)$ is defined by approximating g with smooth functions for which the Föllmer-Itô formula Theorem 2.2 holds (see [FZ06] for similar arguments in a semi-martingale context). Since we are interested in the Riemann sum interpretation of the pathwise integral, we need Theorem 3.6 to make sure that all requirements of Theorem 2.8 are satisfied for typical price paths.

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