Exercises

Exercise 1.1 (2+2 Points)
Define for $p \in [1,\infty]$ the space ℓ^p as

$$\ell^p := \Big\{(x_1, x_2, \dots) : x_k \in \mathbb{R}\,\forall\, k, \|x\|_p := \Big(\sum_{k=1}^{\infty} |x_k|^p\Big)^{1/p} < \infty\Big\}$$

with the interpretation $\|x\|_\infty = \sup_{k\geq 1} |x_k|$ if $p = \infty$. Show that

a) ℓ^p with $d_p(x,y) := \|x-y\|_p$ is a complete metric space;

b) this space is separable if and only if $p < \infty$.

Hint: You may use the Minkowski inequality on $\mathbb{R}^d$:

$$\Big(\sum_{i=1}^{d} |x_i + y_i|^p\Big)^{1/p} \leq \Big(\sum_{i=1}^{d} |x_i|^p\Big)^{1/p} + \Big(\sum_{i=1}^{d} |y_i|^p\Big)^{1/p}.$$

Exercise 1.2 (2 Points)
Let (X,d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X which admits a converging subsequence. Show that then the entire sequence converges.

Exercise 1.3 (2+2+2+2 Points)
Let $\mathcal{C} := C([0,1],\mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$ (both equipped with the Euclidean topology). We write $f_n \xrightarrow{\text{pw}} f$ if $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f, i.e. if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$, and we define

$$\tau := \{U \subseteq \mathcal{C} : \forall f \in U, \forall f_n \xrightarrow{\text{pw}} f\ \exists n_0 \text{ such that } f_n \in U\, \forall n \geq n_0\}.$$

a) Show that τ is a topology.

b) Assume now that d is a metric on $\mathcal{C}$ such that $\lim_{n\to\infty} d(f_n, f) = 0$ if and only if $f_n \xrightarrow{\text{pw}} f$. We define for $x \in [0,1]$

$$\eta(x) := \lim_{k\to\infty} \sup\big\{d(f,0) : \operatorname{supp}(f) \subseteq [x-1/k, x+1/k] \cap [0,1]\big\} \in [0,\infty].$$

Show that there exists $\varepsilon > 0$ with $\eta(x) > \varepsilon$ for uncountably many $x \in [0,1]$.

c) Construct a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ that converges pointwise to 0 but not with respect to d. This contradiction shows that the topology of pointwise convergence on $C([0,1],\mathbb{R})$ is not metrizable.

d) Show that on the other side the pointwise convergence on $C([0,1]\cap\mathbb{Q},\mathbb{R})$ is induced by a metric.

Exercise 1.4 (2+2+2 Points)

Let (X, τ_X) and (Y, τ_Y) be topological spaces. For $A \subseteq X$ we write A' for the set of all *limit points* of A, that is of all $x \in X$ with $x \in \overline{A \setminus \{x\}}$. The *boundary* of A is $\partial A = \overline{A} \setminus A^\circ$. Let $f \colon X \to Y$ be a map. Which ones of the following conditions are necessary respectively sufficient for the continuity of f?

a) We have $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.

b) We have $f(A') \subseteq \overline{f(A)}$ for all $A \subseteq X$.

c) We have $f(\partial A) \subseteq \partial f(A)$ for all $A \subseteq X$.

Due date: Thursday, October 22, 2015.

(You may submit your solutions in groups of two.)