Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 10

Exercises

Exercise 10.1 (5+5 Points)
Define the linear subspace $c_0 \subset \ell^\infty$ via

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} : \lim_{n\to\infty} x_n = 0\}.$$

a) Show that there exists an isometric isomorphism $\Phi\colon \ell^1 \to c_0^\star$.

b) Conclude that every $f \in (\ell^\infty)^\star$ can be written as $f = \Phi(x) + \tilde{f}$, where $x \in \ell^1$ and $\tilde{f}|_{c_0} \equiv 0$ (here we implicitly use that $\Phi(x)$ can be canonically extended from c_0 to ℓ^∞).

Exercise 10.2 (5+5 Points)
The aim of this Exercise is to show that the $\tilde{f}$ of Exercise 10.1 is not necessarily 0. A *Banach limit* is a linear map $f\colon \ell^\infty \to \mathbb{R}$ such that

(i) if $x_n \geq 0$ for all n, then $f((x_n)) \geq 0$;

(ii) $f(Tx) = f(x)$ for all $x \in \ell^\infty$, where $T\colon \ell^\infty \to \ell^\infty$ is the shift operator defined via $(Tx)_n = x_{n+1}$;

(iii) $f(\gamma_1) = 1$, where $(\gamma_1)_n = 1$ for all n.

Show that

a) if f is a Banach limit, then $\liminf_{n\to\infty} x_n \leq f(x) \leq \limsup_{n\to\infty} x_n$ for all $x \in \ell^\infty$ and $f \in (\ell^\infty)^\star$ with $\|f\| = 1$ (in particular, $f|_{c_0} \equiv 0$);

b) a Banach limit exists.

(*Hint:* You will need the Hahn-Banach theorem. The map $\varphi\colon \ell^\infty \to \mathbb{R}$, $\varphi(x) = \limsup_{n\to\infty} x_n$ is convex.)

Happy holidays!

Due date: Thursday, January 7, 2015.
(You may submit your solutions in groups of two.)