Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 11

Exercises

Exercise 11.1 (4 Points)
Let (X_n, d_n), $n \in \mathbb{N}$, be compact metric spaces. Prove that $X = \prod_{n\in\mathbb{N}} X_n$ is compact in the product topology – without using the axiom of choice!

Exercise 11.2 (2 Points)
Let X be a topological space. Recall that $\mathcal{F} \subseteq \mathcal{P}(X)$ has the *finite intersection property* if for all $n \in \mathbb{N}$ and $F_1, \ldots, F_n \in \mathcal{F}$ we have $\bigcap_{k=1}^n F_k \neq \emptyset$. Show that X is compact if and only if for every $\mathcal{F}$ with the finite intersection property the intersection $\bigcap_{F\in\mathcal{F}} \overline{F}$ is non-empty.

Exercise 11.3 (2+4+2 Points)
Consider the space ℓ^p for $p \in [1, \infty]$.

a) Show that for $p \in (1, \infty)$ a sequence $(x^{(n)})_{n\in\mathbb{N}} \subset \ell^p$ converges weakly to $x \in \ell^p$ if and only if $x_i^{(n)} \to x_i$ for all $i \in \mathbb{N}$ and $\sup_n \|x^{(n)}\|_p < \infty$. Construct a sequence $(x^{(n)})_{n\in\mathbb{N}} \subset \ell^p$ that converges weakly but not strongly.

b) Show that for $p = 1$ every sequence $(x^{(n)})_{n\in\mathbb{N}} \subset \ell^1$ that converges weakly to $x \in \ell^1$ also converges strongly to x.

c) Construct for $p = \infty$ a bounded sequence $(f_n)_{n\in\mathbb{N}} \subset (\ell^\infty)^\star$ which has no converging subsequence. Why is this no contradiction to the Banach-Alaoglu theorem?

Exercise 11.4 (2 Points)
Let X be a Hilbert space and $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a bounded sequence. Show that there exists $x \in X$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which converges weakly to x.

Exercise 11.5 (2+2 Points)
Let X be a Banach space and $X^\star$ its dual, and let Y be a normed space.

a) Assume that X is reflexive, $(\ell_n)_{n\in\mathbb{N}} \subset X^\star$ converges to $\ell \in X^\star$ with respect to the weak-* topology, and $(x_n)_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ in the norm topology. Show that $\ell_n(x_n) \to \ell(x)$.

b) Let $A\colon X \to Y$ be a linear map such that for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$

$$x_n \rightharpoonup 0 \quad \Rightarrow \quad Ax_n \rightharpoonup 0.$$

Show that A is a bounded linear operator.

Due date: Thursday, January 14, 2015.
(You may submit your solutions in groups of two.)