Exercises

Exercise 12.1 (2+2 Points)
Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the 1-dimensional torus.

a) Compute the Fourier series of the function

$$f(x) = \begin{cases} -1, & x \in [-1/2, 0), \\ 1, & x \in [0, 1/2). \end{cases}$$

b) Show that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Exercise 12.2 (2+2 Points)
Let $(\alpha(k))_{k\in\mathbb{Z}^d}, (\beta(k))_{k\in\mathbb{Z}^d} \subset \mathbb{C}$ be complex sequences such that $\sum_{k\in\mathbb{Z}^d}(|\alpha(k)| + |\beta(k)|) < \infty$, and let $f(x) = \sum_{k\in\mathbb{Z}^d} \alpha(k)e^{2\pi i k\cdot x}$, $g(x) = \sum_{k\in\mathbb{Z}^d} \beta(k)e^{2\pi i k\cdot x}$ for $x \in \mathbb{T}^d$. Show that

a) for all $k \in \mathbb{Z}^d$

$$\widehat{fg}(k) = \sum_{\ell\in\mathbb{Z}^d} \hat{f}(k-\ell)\hat{g}(\ell);$$

b) for all $x \in \mathbb{T}^d$

$$\sum_{k\in\mathbb{Z}^d} \hat{f}(k)\hat{g}(k)e^{2\pi i k\cdot x} = \int_{\mathbb{T}^d} f(x-y)g(y)\mathrm{d}y.$$

Exercise 12.3 (2+5 Points)
We define $L^2 := L^2(\mathbb{T}^d;\mathbb{C})$ and for $s \geq 0$ the Sobolev space

$$H^s := H^s(\mathbb{T}^d;\mathbb{C}) := \Big\{ u \in L^2 : \|u\|_{H^s}^2 := \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^s |\hat{u}(k)|^2 < \infty \Big\}.$$

a) Show that H^s is a complex Hilbert space with inner product

$$\langle u, v\rangle_{H^s} := \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^s \hat{u}(k)\overline{\hat{v}(k)}.$$

b) Show that for $n \in \mathbb{N}_0$ we have $u \in H^n$ if and only if for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq n$ there exists $\partial^\alpha u \in L^2$ such that

$$\int_{\mathbb{T}^d} \varphi(x)(\partial^\alpha u)(x)\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{T}^d} \partial^\alpha \varphi(x) u(x)\mathrm{d}x.$$

for all $\varphi \in C^\infty := C^\infty(\mathbb{T}^d,\mathbb{C})$, the infinitely smooth 1-periodic functions on $\mathbb{R}^d$. We define $\|u\|_{W^{n,2}} := \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^2}$. Show further that there exists $C > 0$ such that for all $u \in H^n$

$$\frac{1}{C}\|u\|_{W^{n,2}} \leq \Big(\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^n |\hat{u}(k)|^2 \Big)^{1/2} \leq C\|u\|_{W^{n,2}}.$$

Exercise 12.4 (5 Points)

Let H^s be as in Exercise 12.3 and write $\Delta u := \sum_{j=1}^{d} \frac{\partial^2}{\partial_{x_j}^2} u$ for $u \in H^2$. Let $F \in C_b^1(\mathbb{R}, \mathbb{R})$, i.e. both F and its derivative are globally bounded, and let $a > \|F'\|_\infty$. Show that there exists a unique solution $u \in H^2$ to the nonlinear PDE

$$(a - \Delta)u = F(u).$$

(*Hint:* Start by performing a Picard iteration in H^0.)

Due date: Thursday, January 21, 2015.

(You may submit your solutions in groups of two.)