Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 13

Exercises

Exercise 13.1 (4+2 Points)
Let $u \in C_c^\infty(\mathbb{R}, \mathbb{C})$ with support contained in $[-R, R]$ for some $R > 0$.

a) Show that $\hat{u}\colon \mathbb{R} \to \mathbb{C}$ can be extended to a holomorphic function on all of $\mathbb{C}$ such that for all $n \in \mathbb{N}$ there exists $C_n > 0$ (depending on u) with

$$|\hat{u}(\xi)| \leq C_n (1 + |\xi|)^{-n} e^{2\pi |\mathrm{Im}(\xi)| R}, \qquad \xi \in \mathbb{C}.$$

b) Conclude that if $\hat{u}$ is compactly supported (as a function on $\mathbb{R}$), then $u \equiv 0$.

Exercise 13.2 (4+2 Points)
Let $u \in \mathscr{S}(\mathbb{R})$, the Schwarz functions on $\mathbb{R}$.

a) Prove the Heisenberg uncertainty principle:

$$\Big(\int_R |x|^2 |u(x)|^2 \mathrm{d}x \Big)^{1/2} \Big(\int_\mathbb{R} |\xi|^2 |\hat{u}(\xi)|^2 \mathrm{d}x \Big)^{1/2} \geq \frac{1}{4\pi} \|u\|_{L^2}^2.$$

(*Hint:* You may use Plancherel's formula $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$ which we will prove in the next lecture.)

b) Show that equality holds if and only if u is Gaussian, that is if there exist $C, \lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) < 0$ such that $u(x) = Ce^{\lambda x^2}$.

Exercise 13.3 (4+4 Points)
Consider the wave equation for $u \in C([0,\infty), \mathscr{S}(\mathbb{R})) \cap C^2([0,\infty), C_b(\mathbb{R}, \mathbb{C}))$ and $c > 0$,

$$\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u. \tag{1}$$

We assume that $\int_\mathbb{R} u(t,x)\mathrm{d}x$ does not depend on $t \geq 0$.

a) Show that there exist $f, g \in \mathscr{S}$ such that

$$u(t,x) = f(x + ct) + g(x - ct)$$

for all $t \geq 0$ and $x \in \mathbb{R}$.

(*Hint:* Apply the Fourier transform to the space variable x.)

b) Show that given $\varphi, \psi \in \mathscr{S}(\mathbb{R})$ there exists exactly one $u \in C([0,\infty), \mathscr{S}(\mathbb{R})) \cap C^2([0,\infty), C_b(\mathbb{R}, \mathbb{C}))$ which solves (1) and satisfies $u(0) = \varphi$, $\frac{\partial}{\partial t} u(0) = \psi'$.

Due date: Thursday, January 28, 2015.
(You may submit your solutions in groups of two.)