Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 2

Exercises

Exercise 2.1 (4 Points)
Let $(X, \|\cdot\|)$ be a normed $\mathbb{K}$-vector space and let U be a finite-dimensional linear subspace of X (i.e. $U \subseteq X$ is such that $\alpha x + \beta y \in U$ for all $x, y \in U$, $\alpha, \beta \in \mathbb{K}$ and U has a finite basis). Show that U is complete and in particular closed.

Exercise 2.2 (4 Points)
Let $(X, \|\cdot\|)$ be an infinite-dimensional $\mathbb{K}$-Banach space. Show that every basis of X is uncountable.

Hint: Use Baire's theorem.

Exercise 2.3 (4 Points)
Let (X, d) be a non-empty compact metric space and let $f\colon X \to X$ be such that

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x \neq y \in X.$$

Show that f admits a unique fixpoint.

Hint: Consider the function $X \ni x \mapsto d(x, f(x)) \in [0, \infty)$.

Exercise 2.4 (4+4 Points)
Consider a non-empty compact interval $[a, b] \subset \mathbb{R}$. The space of continuous functions from $[a, b]$ to $\mathbb{R}$ is denoted by $C([a, b], \mathbb{R})$. For $(n, \alpha) \in \mathbb{N}_0 \times [0, 1]$ we define

$$C^{n,\alpha}([a, b], \mathbb{R}) := \Big\{ f \in C([a, b], \mathbb{R}) : f \text{ is } n \text{ times continuously differentiable and } \|f^{(n)}\|_\alpha < \infty \Big\},$$

where

$$\|f\|_\alpha := \sup_{\substack{x,y \in [a,b] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

a) Show that
$$C^{1,0}([a, b], \mathbb{R}) \subseteq C^{0,\beta}([a, b], \mathbb{R}) \subseteq C^{0,\alpha}([a, b], \mathbb{R}) \subseteq C([a, b], \mathbb{R})$$
whenever $0 \le \alpha \le \beta \le 1$, and that for $0 < \alpha < \beta \le 1$ all inclusions are strict.

b) Show that for $\alpha \in (0, 1]$ the space $C^{0,\alpha}([a, b], \mathbb{R})$ equipped with the norm $\|f\| := \|f\|_\alpha + |f(a)|$ is not separable and $(C^{0,\alpha}([a, b], \mathbb{R}), \|\cdot\|_\infty)$ is not complete, where $\|f\|_\infty := \max_{x \in [a,b]} |f(x)|$.

Due date: Thursday, October 29, 2015.
(You may submit your solutions in groups of two.)