Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 3

Exercises

Exercise 3.1 (3+3 Points)
Let $(X, \|\cdot\|)$ be a normed space and let $U \neq X$ be a closed linear subspace of X.

a) Show that for every $\delta > 0$ there exists $x_\delta \in X$ with $\|x_\delta\| = 1$ and

$$d(x_\delta, U) = \inf\{\|x_\delta - y\| : y \in U\} \geq 1 - \delta.$$

Hint: Show that there is an $x \in X$ with $d(x, U) > 0$. Deduce that there must exist $u \in U$ with $\|x - u\| < d(x, U)/(1 - \delta)$.

b) Show that if every bounded sequence in X has a converging subsequence, then the dimension of X is finite. This completes the proof of Theorem 1.38 from the lecture.

Exercise 3.2 (4 Points)
Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and let $K \subseteq X$ be compact. Show that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with

$$\Big(\sum_{k=n}^{\infty} |\langle x, e_k \rangle|^2 \Big)^{1/2} < \varepsilon$$

for all $x \in K$.

Exercise 3.3 (3+3 Points)
Let $(X, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space on which the parallelogram identity holds, that is

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all $x, y \in X$. Define

$$\langle x, y \rangle := \frac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2\big), \qquad x, y \in X.$$

a) Show that for all $x, y, z \in X$

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$$

b) Show that for all $\lambda \in \mathbb{R}$ and $x, y \in X$

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle.$$

Deduce that $\langle \cdot, \cdot \rangle$ is an inner product on X with $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$.

Hint: First show the identity for $\lambda \in \mathbb{N}$, then for $\lambda \in \mathbb{Z}$, and then for $\lambda \in \mathbb{Q}$.

Exercise 3.4 (4 Points)
Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Show that the unit ball $B[0, 1]$ is a closed convex subset of X and calculate the projection $P(x)$ of an arbitrary $x \in X$ onto $B[0, 1]$.

Due date: Thursday, November 5, 2015.
(You may submit your solutions in groups of two.)